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Complex Variables

In-Class Exercise Solutions

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NAME:
Consider the function

$$f(z) = e^{z^2}$$

1. **Recalling that** $(x+iy)^2 = (x^2-y^2)+i2xy$**, write** $f(x+iy) = \rho(x,y)e^{i\phi(x,y)}$ **where** $\rho(x,y)$ **and** $\phi(x,y)$ **are real functions of** x **and** y**.**

   Since $z^2 = (x+iy)^2 = (x^2-y^2)+i2xy$ we have

   $$f(x+iy) = e^{(x+iy)^2} = e^{(x^2-y^2)+i2xy} = e^{x^2-y^2}e^{i2xy}$$

   $$\rho(x,y) = \boxed{e^{x^2-y^2}} \qquad \phi(x,y) = \boxed{2xy}$$

2. **Use the Euler relation** $e^{i\alpha} = \cos\alpha + i\sin\alpha$ **and the results of part 1 to write** $f(x+iy) = u(x,y)+iv(x,y)$**, where** $u(x,y)$ **and** $v(x,y)$ **are real functions of** x **and** y**.**

   The Euler relation tells us $e^{i2xy} = \cos 2xy + i\sin 2xy$, so

   $$f(x+iy)e^{x^2-y^2}e^{i2xy} = e^{x^2-y^2}(\cos 2xy + i\sin 2xy) = e^{x^2-y^2}\cos 2xy + ie^{x^2-y^2}\sin 2xy$$

   $$u(x,y) = \boxed{e^{x^2-y^2}\cos 2xy} \qquad v(x,y) = \boxed{e^{x^2-y^2}\sin 2xy}$$

3. Take the partial derivatives of the $u(x, y)$ and $v(x, y)$ you found in part 2.

The partial derivatives of $u(x, y) = e^{x^2-y^2} \cos 2xy$ and $e^{x^2-y^2} \sin 2xy$ are

$$\frac{\partial u}{\partial x} = (2xe^{x^2-y^2}) \cos 2xy + e^{x^2-y^2}(-2y \sin 2xy) = 2e^{x^2-y^2}(x \cos 2xy - y \sin 2xy)$$
$$\frac{\partial u}{\partial y} = (-2ye^{x^2-y^2}) \cos 2xy + e^{x^2-y^2}(-2x \sin 2xy) = -2e^{x^2-y^2}(y \cos 2xy + x \sin 2xy)$$
$$\frac{\partial v}{\partial x} = (2xe^{x^2-y^2}) \sin 2xy + e^{x^2-y^2}(2y \cos 2xy) = 2e^{x^2-y^2}(y \cos 2xy + x \sin 2xy)$$
$$\frac{\partial v}{\partial y} = (-2ye^{x^2-y^2}) \sin 2xy + e^{x^2-y^2}(2x \cos 2xy) = 2e^{x^2-y^2}(x \cos 2xy - y \sin 2xy)$$

$$\frac{\partial u}{\partial x} = \boxed{2e^{x^2-y^2}(x \cos 2xy - y \sin 2xy)} \qquad \frac{\partial v}{\partial x} = \boxed{2e^{x^2-y^2}(y \cos 2xy + x \sin 2xy)}$$

$$\frac{\partial u}{\partial y} = \boxed{-2e^{x^2-y^2}(y \cos 2xy + x \sin 2xy)} \qquad \frac{\partial v}{\partial y} = \boxed{2e^{x^2-y^2}(x \cos 2xy - y \sin 2xy)}$$

4. Use the results of part 3 to show $f(z) = e^{z^2}$ is analytic everywhere.

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, the Cauchy-Riemann equations are satisfied for all x and y. This means $f(z)$ is differentiable everywhere, which means it's analytic everywhere.