Conformal Mappings
(Zill & Wright Chapter 20)

1016-420-02: Complex Variables*

Winter 2012-2013

Contents

Tuesday 5 February 2012

1 Motivation

We turn now from contour integration back to some ideas we first touched on when we introduced complex functions, specifically:

- A complex function $w = f(z)$ can be thought of as a mapping from the $z = x + iy$ plane with coördinates (x, y) to the $w = u + iv$ plane with coördinates (u, v).
- If $\varphi(x, y)+i\psi(x, y)$ is analytic, then the real functions $\varphi(x, y)$ and $\psi(x, y)$ are harmonic, i.e., each obeys the Laplace equation
$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \tag{1.1}$$

The main application of this will be in converting the Laplace equation with boundary conditions on an awkward region in the (x, y) plane to one on a more convenient region in the (u, v) plane. I.e., if we have a harmonic function $U(u, v)$ which is part of an analytic function
$$W = F(w) = U(u, v) + iV(u, v) \tag{1.2}$$
on some convenient region in the w plane and $w = f(z) = u(x, y) + iv(x, y)$ is an analytic function which maps the region of interest in the (x, y) plane onto the convenient region in

the (u, v) plane, then the chain rule tells us that $F(f(z))$ is an analytic function of z; that means if we write

$$\varphi(x, y) + i\psi(x, y) := F(f(z)) = U(u(x, y), v(x, y)) + iV(u(x, y), v(x, y)) \tag{1.3}$$

then we can see that

$$\varphi(x, y) = U(u(x, y), v(x, y)) \tag{1.4}$$

is a harmonic function of x and y with boundary conditions on the appropriate region.

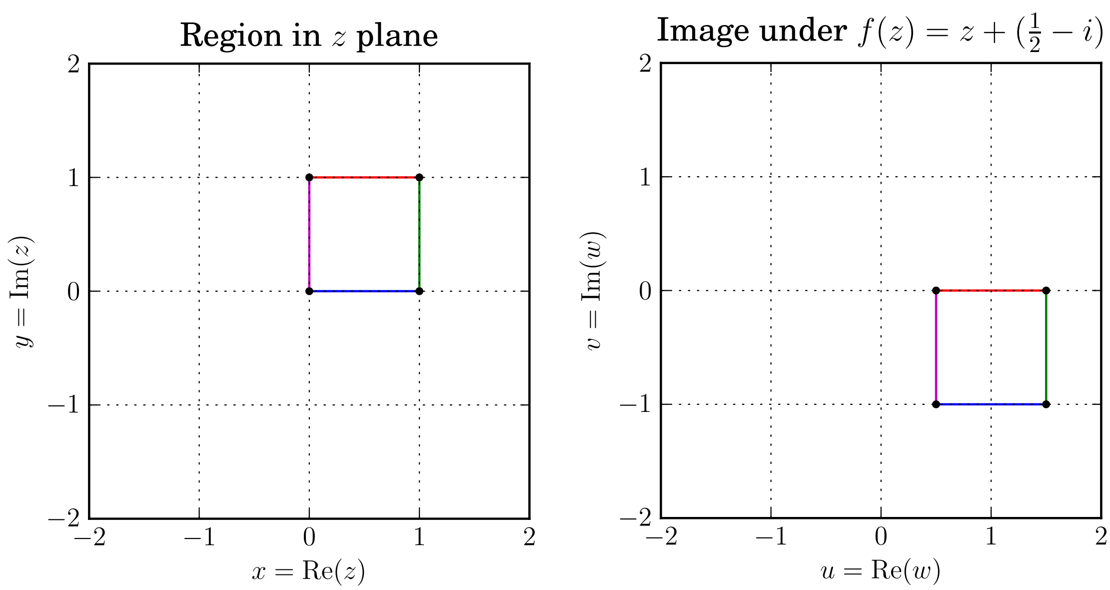
2 Complex Functions as Mappings

Recall that when we first introduced complex functions, we proposed the idea of mapping the z plane, with coördinates $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$, into the w plane, with coördinates $u = \mathrm{Re}(w)$ and $v = \mathrm{Im}(w)$.

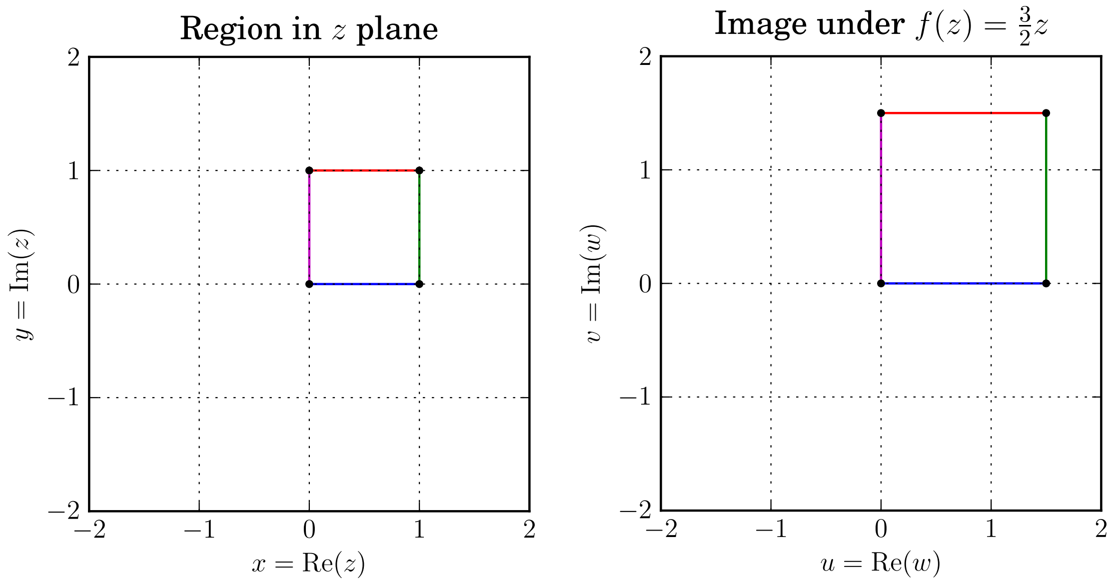
We now want to think about how this mapping affects not just curves, but regions of the (x, y) plane.

Translation: $f(z) = z + b$ where $b = h + ik$; then

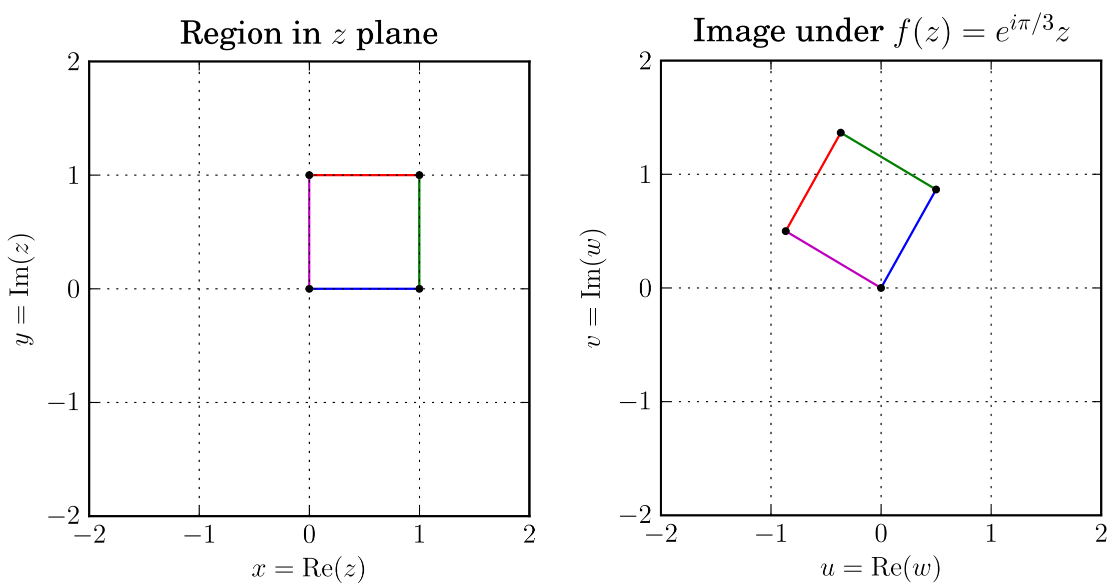
$$u = x + h \tag{2.1a}$$
$$v = y + k \tag{2.1b}$$



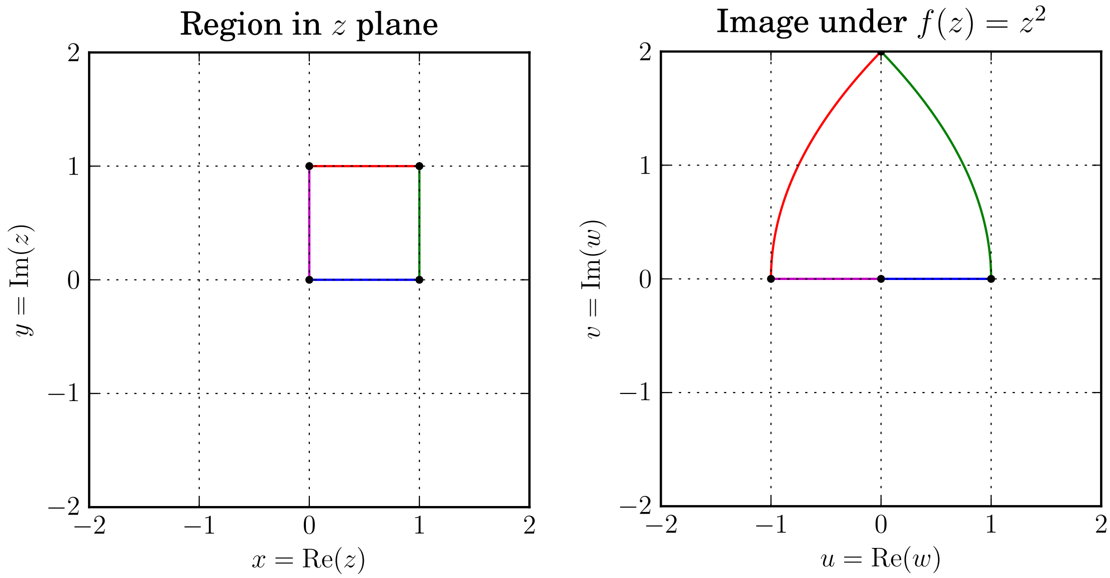
Magnification: $f(z) = \alpha z$ where α is a positive real constant.



Rotation: $f(z) = e^{i\theta_0} z$ where θ_0 is a real constant.



Effect of raising to a power: $f(z) = z^\alpha$ scales angles at $z = 0$ by a factor of α:



Practice Problems

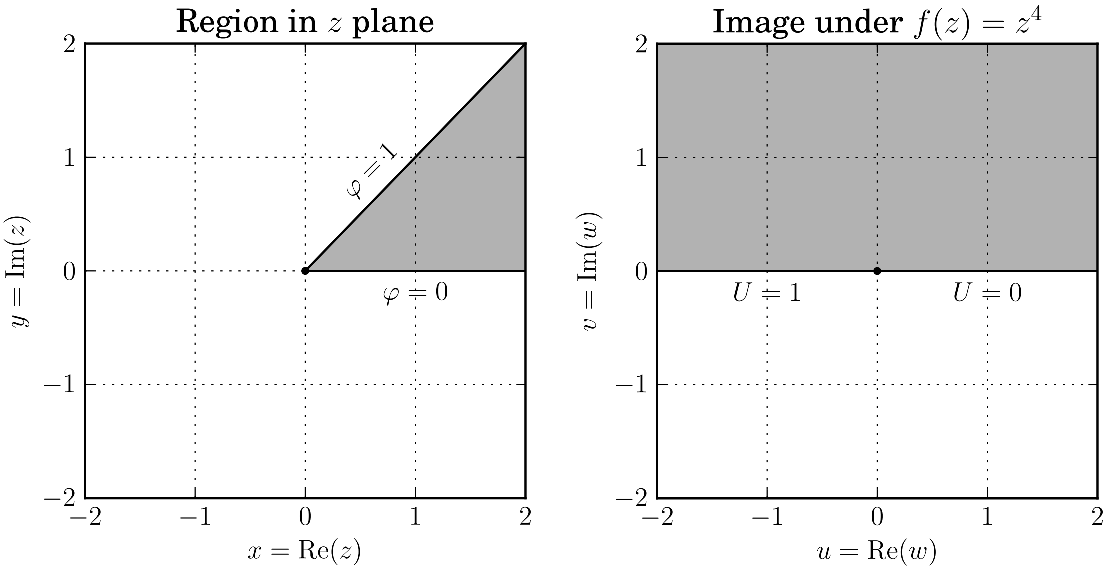
20.1.1, 20.1.5, 20.1.7, 20.1.11, 20.1.13, 20.1.15, 20.1.21, 20.1.23, 20.1.25, 20.1.29

Thursday 7 February 2013

3 Conformal Mappings

3.1 Application to Dirichlet Problems: Exercise 20.2.19

We are asked to solve the Dirichlet problem (Laplace equation with specified boundary conditions for a region which looks like the figure at left:



We are also told to use the function $U = (1/\pi)\,\mathrm{Arg}\,w$ to do this. Now if we remember that $\mathrm{Ln}\,w = \log_e|w| + i\,\mathrm{Arg}\,w$, then we can see that the proposed U is part of the analytic function

$$\begin{aligned} F(w) &= \frac{1}{i\pi}\,\mathrm{Ln}\,w = \frac{1}{\pi}\,\mathrm{Arg}\,w - \frac{i}{\pi}\log_e|w| = \frac{1}{\pi}\,\mathtt{atan2}(v,u) + i\left(-\frac{1}{\pi}\log_e\sqrt{u^2+v^2}\right) \\ &= U(u,v) + iV(u,v) \end{aligned} \tag{3.1}$$

If we note that

$$\mathtt{atan2}(v,0) = \begin{cases} 0 & \text{if } v > 0 \\ \pi & \text{if } v < 0 \end{cases} \tag{3.2}$$

we see that $U(u=0, v>0) = 0$ and $U(u=0, v<0) = 1$, so $U(u,v)$ solves the Dirichlet problem with the boundary conditions shown on the right in the figure above. So if we have a conformal transformation which maps the (x,y) region above (a 45° wedge) onto the (u,v) region above (the upper half plane), we can use it to solve the Dirichlet problem in the (x,y) plane. We can do this if we quadruple the angle at $z=0$, and we saw last time that the mapping which does this is $f(z) = z^4$. So the solution $\varphi(x,y)$ can be obtained from

$$\varphi(x,y) + i\psi(x,y) = F(f(z)) = \frac{1}{i\pi}\,\mathrm{Ln}(z^4) \tag{3.3}$$

Now, we could expand out $(x+iy)^4$, but since it's inside a logarithm, it's much simpler to note $\mathrm{Ln}(z^4) = 4\,\mathrm{Ln}\,z$ so that

$$\varphi(x,y) + i\psi(x,y) = \frac{1}{i\pi}\,\mathrm{Ln}(z^4) = \frac{4}{i\pi}\,\mathrm{Ln}\,z = \frac{4}{\pi}\,\mathrm{Arg}\,z - \frac{4i}{\pi}\log_e|z| \tag{3.4}$$

and thus the solution is

$$\varphi(x,y) = \frac{4}{\pi}\,\mathtt{atan2}(y,x) \tag{3.5}$$

Now in this case, since the wedge contains only $x > 0$, we can write the solution to the Dirichlet problem as

$$\varphi(x,y) = \frac{4}{\pi}\tan^{-1}\frac{y}{x} \qquad x \geq y \geq 0 \tag{3.6}$$

Note that we didn't actually need to use a conformal transformation to solve this problem, because the region in the (x, y) plane was simple enough that we could have written down $\frac{4}{\pi}\tan^{-1}\frac{y}{x}$ in the first place. But in problems with more complicated geometry, more complicated conformal transformations are needed. There is a table of these in Appendix IV of Zill and Wright.

3.2 Conditions for Conformality

Having considered complex functions as mappings from the (x, y) plane to the (u, v) plane, we now turn specifically to *conformal mappings*. "Conformal" means preserving angles, i.e., if two curves $z_1(t)$ and $z_2(t)$ intersect (at a point $z_0 = z_1(t_0) = z_2(t_0)$) at an angle α in the (x, y) plane, the mapped curves $w_1(t) = f(z_1(t))$ and $w_2(t) = f(z_2(t))$ will intersect (at the point $f(z_0) = w_1(t_0) = w_2(t_0)$) at the same angle in the (u, v) plane. We can see what conditions on $f(z)$ achieve this by considering the heading of a curve $z(t)$, i.e., what angle its tangent vector makes to a line parallel to the real axis. Since the curve $z(t) = x(t)+iy(t)$ has x coördinate $x(t)$ and y coördinate $y(t)$, its tangent vector is

$$x'(t)\hat{x} + y'(t)\hat{y} \tag{3.7}$$

The heading of this vector is

$$\texttt{atan2}(y'(t), x'(t)) = \operatorname{Arg} z'(t) \tag{3.8}$$

so the angle at which the curves $z_1(t)$ and $z_2(t)$ intersect is the difference of their headings,

$$\alpha = \arg z_2'(t_0) - \arg z_1'(t_0) \tag{3.9}$$

where we have written arg rather than the principal value Arg because adding multiples of 2π to won't change its physical meaning.

Likewise, heading of the curve $w(t) = f(z(t))$ is $\operatorname{Arg} w'(t)$. The chain rule tells us that

$$w'(t) = f'(z(t))z'(t) \tag{3.10}$$

so the heading is

$$\arg w'(t) = \arg f'(z(t)) + \arg z'(t) \tag{3.11}$$

where we have used the fact that

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \tag{3.12}$$

This means that the headings of the two curves $w_1(t)$ and $w_2(t)$ at the point of intersection are

$$\arg w_1'(t_0) = \arg f'(z_0) + \arg z_1'(t_0) \tag{3.13a}$$
$$\arg w_2'(t_0) = \arg f'(z_0) + \arg z_2'(t_0) \tag{3.13b}$$

which means the angle between them is

$$\arg w_2'(t_0) - \arg w_1'(t_0) = \arg z_2'(t_0) - \arg z_1'(t_0) \tag{3.14}$$

This demonstration works as long as $\arg f'(z_0)$ is well defined. So the function $f(z)$ has to be analytic (so that $f'(z)$ exists along the curves), but also $f'(z_0)$ must be non-zero, because $\arg 0$ is undefined. I.e.,

$$f(z) \text{ defines a conformal mapping wherever } f(z) \text{ is analytic and } f'(z) \neq 0 \tag{3.15}$$

Note that this was the case with our example of $f(z) = z^2$ on Tuesday. Since $f'(z) = 2z$ which is zero only at $z = 0$, the mapping is conformal except at $z = 0$. We saw this, as the right angles at the corners of the square remained right angles, except for the one at the origin.

4 Geometric Applications of Analytic Functions

4.1 Potential for a Vector Field

Recall that when we first defined a complex function

$$f(z) = f(x+iy) = u(x,y) + iv(x,y) \tag{4.1}$$

we deduced the Cauchy-Riemann equations from

$$f'(z) = \frac{\partial}{\partial x} f(x+iy) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1}{i}\frac{\partial}{\partial y} f(x+iy) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \tag{4.2}$$

We also saw that the Cauchy-Riemann equations meant that the Pólya vector field

$$\vec{H} = u(x,y)\hat{x} - v(x,y)\hat{y} \tag{4.3}$$

had zero divergence and zero curl:

$$\operatorname{div} \vec{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \tag{4.4a}$$
$$\operatorname{curl} \vec{H} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{4.4b}$$

Another fact from vector calculus is that any vector field with zero curl can be written as a gradient of some scalar field, i.e., there should be a $\varphi(x, y)$ such that

$$\vec{H} = \tilde{\nabla}\,\varphi = \frac{\partial \varphi}{\partial x}\hat{x} + \frac{\partial \varphi}{\partial y}\hat{y} \tag{4.5}$$

We can construct this field by starting with the analytic function $f(z)$. Since it is analytic, we can construct an antiderivative $F(z)$ such that $f(z) = F'(z)$. Write the real and imaginary parts of the antiderivative as

$$F(z) = F(x + iy) = \varphi(x, y) + i\psi(x, y) \tag{4.6}$$

We note that we can write the derivative as

$$F'(z) = \frac{\partial}{\partial x}F(x + iy) = \frac{\partial \varphi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{1}{i}\frac{\partial}{\partial y}F(x + iy) = \frac{\partial \psi}{\partial y} - i\frac{\partial \varphi}{\partial y} \tag{4.7}$$

The Cauchy-Riemann equations applied to $F(z)$ are

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \tag{4.8a}$$

$$\frac{\partial \psi}{\partial x} = -\frac{\partial \varphi}{\partial y} \tag{4.8b}$$

so we can also write

$$f(z) = F'(z) = \frac{\partial \varphi}{\partial x} - i\frac{\partial \varphi}{\partial y} \tag{4.9}$$

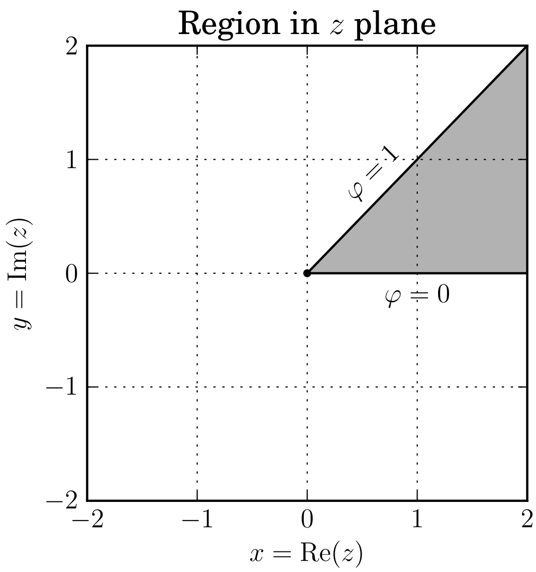
That makes the Pólya vector field for $f(z)$

$$\vec{H} = \frac{\partial \varphi}{\partial x}\hat{x} + \frac{\partial \varphi}{\partial y}\hat{y} = \tilde{\nabla}\,\varphi \tag{4.10}$$

The scalar field φ, which is harmonic, is called the potential for the Pólya vector field $\vec{H}$.

4.2 Sample Application: Exercise 20.6.5

We are told that the potential on the wedge $0 \le \operatorname{Arg} z \le \pi/4$ satisfies the boundary conditions $\varphi(x,0) = 0$ and $\varphi(x,x) = 1$ for $x > 0$. We're asked to find the complex potential, equipotential lines, and force field $\vec{F}$. This is just the geometry we considered in Exercise 20.2.1:



so we know the complex potential is

$$F(z) = \varphi(x,y) + i\psi(x,y) = \frac{4}{i\pi}\operatorname{Ln} z = \frac{4}{\pi}\,\mathtt{atan2}(y,x) - \frac{4i}{\pi}\log_e\sqrt{x^2+y^2} \tag{4.11}$$

The derivative of this is

$$f(z) = F'(z) = \frac{4}{i\pi z} = \frac{4}{\pi}\frac{1}{-y+ix}\frac{-y-ix}{-y-ix} = \frac{4}{\pi}\frac{-y-ix}{x^2+y^2} \tag{4.12}$$

and the Pólya vector field is

$$\vec{H} = \frac{4}{\pi}\frac{-y-ix}{x^2+y^2}\left(-\frac{y}{x^2+y^2}\,\hat{x} + \frac{x}{x^2+y^2}\,\hat{y}\right) \tag{4.13}$$

Note that since

$$\vec{H} = \frac{\partial\varphi}{\partial x}\hat{x} + \frac{\partial\varphi}{\partial y}\hat{y} = \tilde{\nabla}\,\varphi \tag{4.14}$$

this tells us that

$$\frac{\partial}{\partial x}\,\mathtt{atan2}(y,x) = -\frac{y}{x^2+y^2} = \frac{\partial\theta}{\partial x} \tag{4.15a}$$

$$\frac{\partial}{\partial y}\,\mathtt{atan2}(y,x) = \frac{x}{x^2+y^2} = \frac{\partial\theta}{\partial y} \tag{4.15b}$$

By the way, I think this means the answer to this problem in the back of Zill and Wright is actually wrong!

Practice Problems

20.2.1, 20.2.3, 20.2.5, 20.2.7, 20.2.9, 20.2.19, 20.2.23, 20.6.1, 20.6.3, 20.6.5