

Using Geometry to Convert Complex Numbers Between Polar & Cartesian Form

1016-420-02: Complex Variables*

Winter 2012-2013

We've seen how to convert the complex number $z = x + iy = re^{i\theta}$ between Cartesian coordinates (x, y) and polar coordinates (r, θ) using the transformations

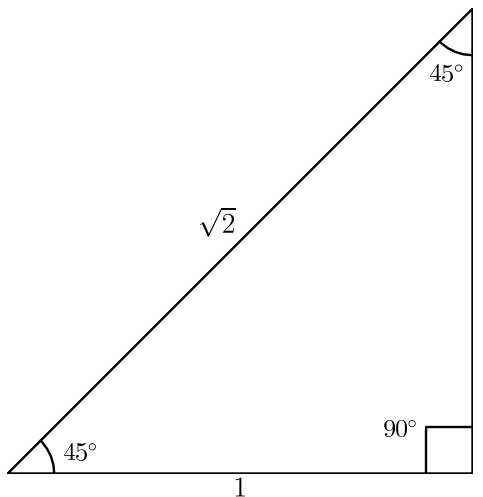
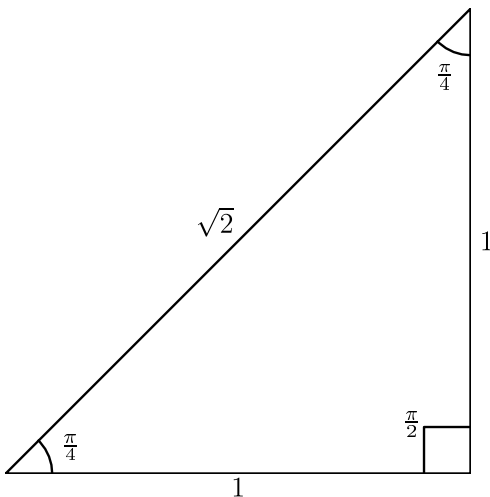
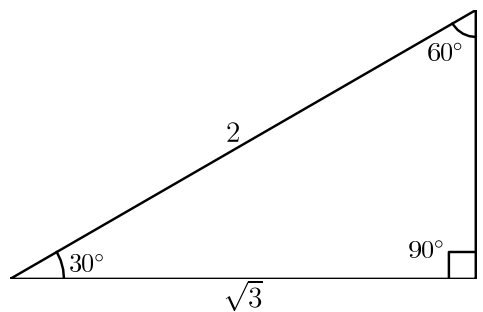
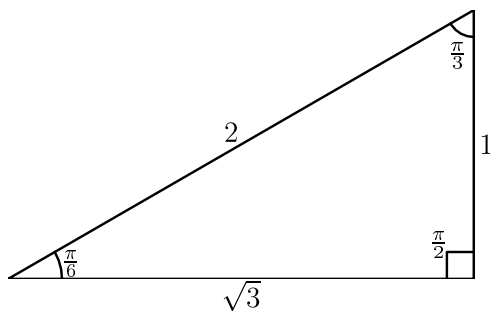
$$x = r \cos \theta \quad (1a)$$

$$r = \sqrt{x^2 + y^2} \quad (2a)$$

$$y = r \sin \theta \quad (1b)$$

$$\theta = \text{atan2}(y, x) \quad (2b)$$

together with a table of trig functions of multiples of $\pi/4$ and $\pi/6$. Rather than memorizing or reconstructing that table of trig functions, one can instead remember the following two triangles (shown with their angles labelled in radians, and also for reference in degrees):



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Examples

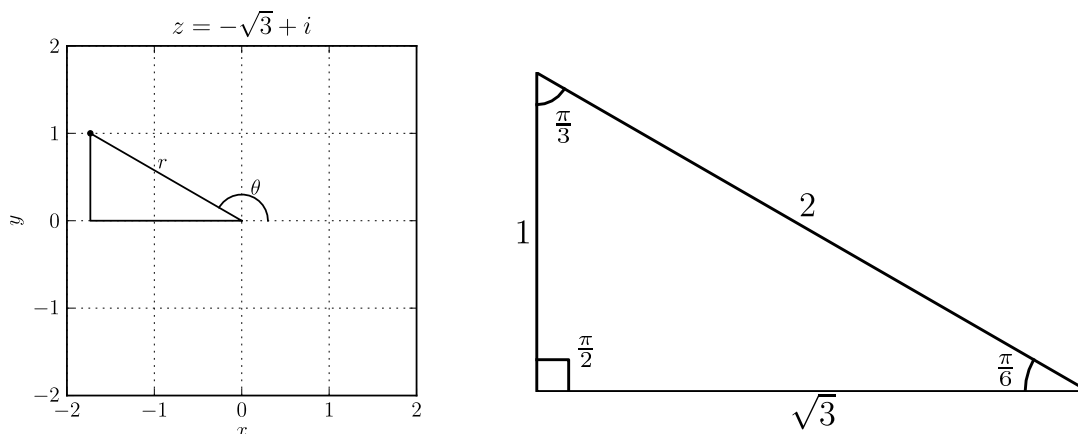
1. Convert $z = -\sqrt{3} + i$ into polar form.

First we read off from $z = x + iy = -\sqrt{3} + i$ the Cartesian coordinates

$$x = -\sqrt{3} \tag{3a}$$

$$y = 1 . \tag{3b}$$

This point lies in the second quadrant:



The sides $\sqrt{3}$ and 1 fit into our 30° – 60° – 90° triangle, which we blow up in the right-hand side of the diagram. The modulus is $r = |z| = \sqrt{x^2 + y^2} = \sqrt{3 + 1} = 2$, which can also be seen from the fact that the hypotenuse of the triangle is 2. We see that the angle θ is between $\pi/2$ (90°) and π (180°). Since the small angle of the triangle is $\pi/6$, we must have

$$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6} , \tag{4}$$

i.e.,

$$z = -\sqrt{3} + i = 2e^{i5\pi/6} . \tag{5}$$

Note that these triangles tell us that

$$\cos \theta = \frac{x}{r} = \cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} \tag{6a}$$

$$\sin \theta = \frac{y}{r} = \sin \frac{5\pi}{6} = \frac{1}{2} . \tag{6b}$$

Note also that we would have got the wrong answer if we'd taken

$$\tan^{-1} \left(\frac{y}{x} \right) = \tan^{-1} \left(\frac{1}{-\sqrt{3}} \right) = \tan^{-1} \left(-\frac{1}{\sqrt{3}} \right) = -\frac{\pi}{6} \neq \theta . \tag{7}$$

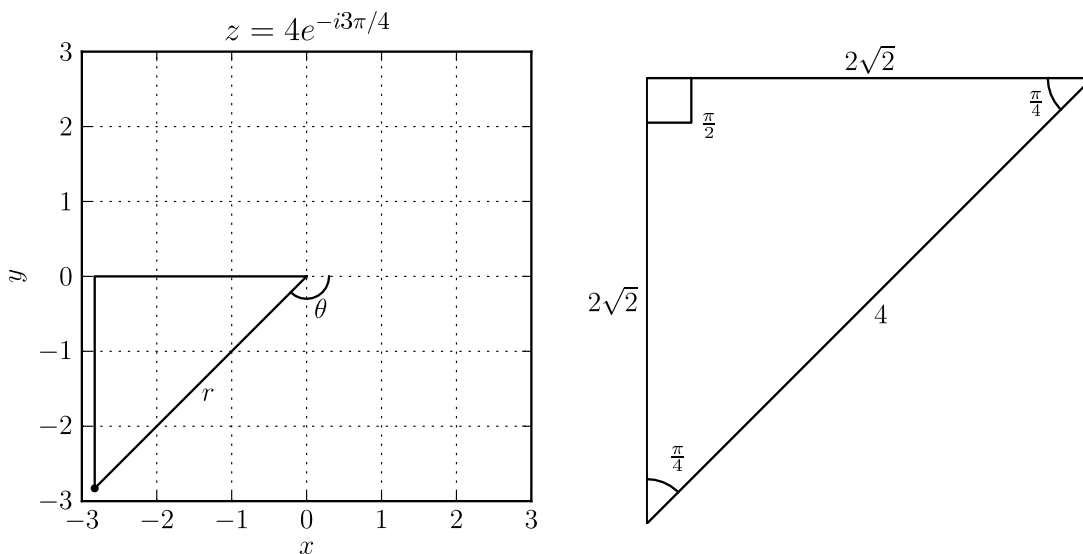
2. Convert $z = 4e^{-i3\pi/4}$ into Cartesian form.

First we read off from $z = re^{i\theta} = 4e^{-i3\pi/4}$ the polar coordinates

$$r = 4 \tag{8a}$$

$$\theta = -\frac{3\pi}{4} . \tag{8b}$$

Since $\theta = -3\pi/4$ is between $-\pi$ (-180°) and $-\pi/2$ (-90°), the point lies in the third quadrant:



Looking at the diagram and using $\pi - 3\pi/4 = \pi/4$, we see that the angles in the triangle are 45° , which gives us the isosceles right triangle blown up on the right. We've scaled up the triangle by a factor of $2\sqrt{2}$ so that the hypotenuse is $r = 4$, from which we can read off

$$x = -2\sqrt{2} \tag{9a}$$

$$y = -2\sqrt{2} , \tag{9b}$$

i.e.,

$$z = 4e^{-i3\pi/4} = -2\sqrt{2} - i2\sqrt{2} . \tag{10}$$

Note that these triangles tell us that

$$\cos \theta = \frac{x}{r} = \cos \left(-\frac{3\pi}{4} \right) = -\frac{\sqrt{2}}{2} \tag{11a}$$

$$\sin \theta = \frac{y}{r} = \sin \left(-\frac{3\pi}{4} \right) = -\frac{\sqrt{2}}{2} . \tag{11b}$$

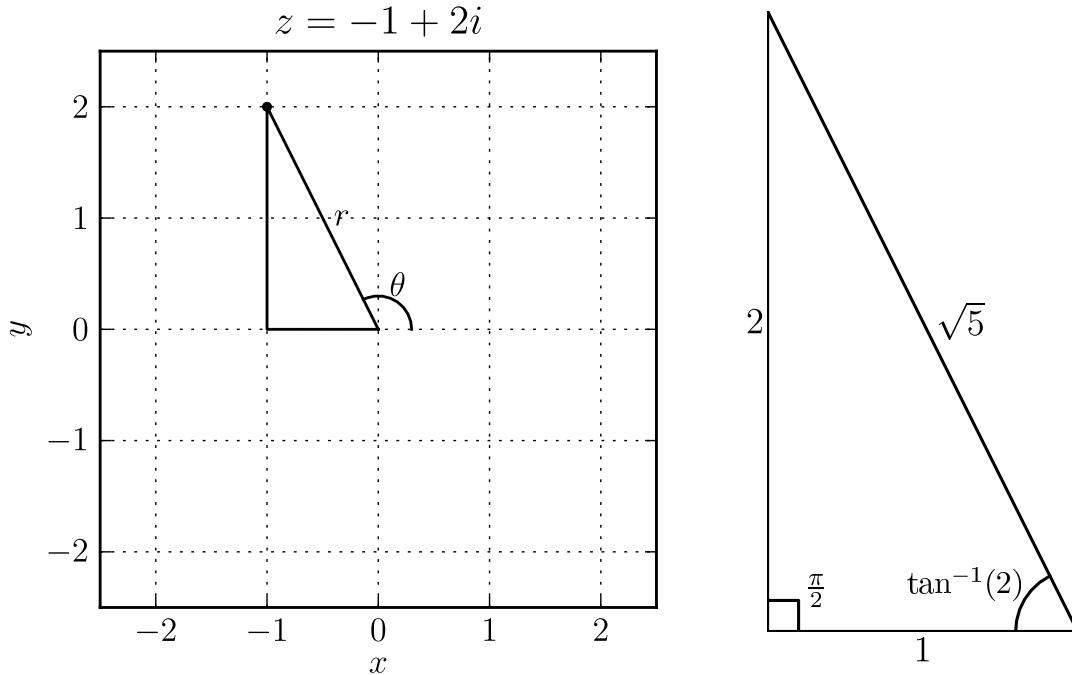
3. Convert $z = -1 + 2i$ into polar form.

The geometric method also works if the angles are not multiples of $\pi/6$ or $\pi/4$. For this example, we know $\theta = \text{atan2}(2, -1)$, but if we don't have the `atan2` function available to us, we can look at the triangle. First we read off from $z = x + iy = -1 + 2i$ the Cartesian coördinates

$$x = -1 \tag{12a}$$

$$y = 2 . \tag{12b}$$

This point again lies in the second quadrant:



Now the sides 1 and 2 don't fit into either of our triangles, but we can still use trigonometry to work out the angles. First, the modulus is $r = |z| = \sqrt{x^2 + y^2} = \sqrt{1 + 4} = \sqrt{5}$, which is also the hypotenuse of the triangle. We see that the angle θ is between $\pi/2$ and π . The large angle of the triangle is $\pi - \theta$, and trigonometry tells us that $\tan(\pi - \theta) = 2$ (and likewise that $\sin(\pi - \theta) = 2/\sqrt{5}$ and $\cos(\pi - \theta) = 1/\sqrt{5}$). Thus $\pi - \theta = \tan^{-1}(2) \approx 1.10715$ and

$$\theta = \pi - \tan^{-1}(2) \approx 2.03444 , \tag{13}$$

i.e.,

$$z = -1 + 2i = \sqrt{5} e^{i[\pi - \tan^{-1}(2)]} \approx 2.23607 e^{i2.03444} . \tag{14}$$

Note that we can also check

$$x \approx 2.23607 \cos(2.03444) \approx -1 \tag{15a}$$

$$y \approx 2.23607 \sin(2.03444) \approx 2 . \tag{15b}$$

Note also that we would have got the wrong answer if we'd taken

$$\tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2}{-1}\right) = \tan^{-1}(-2) = -\tan^{-1}(2) \neq \theta . \tag{16}$$