Using Geometry to Convert Complex Numbers Between Polar & Cartesian Form

1016-420-02: Complex Variables*

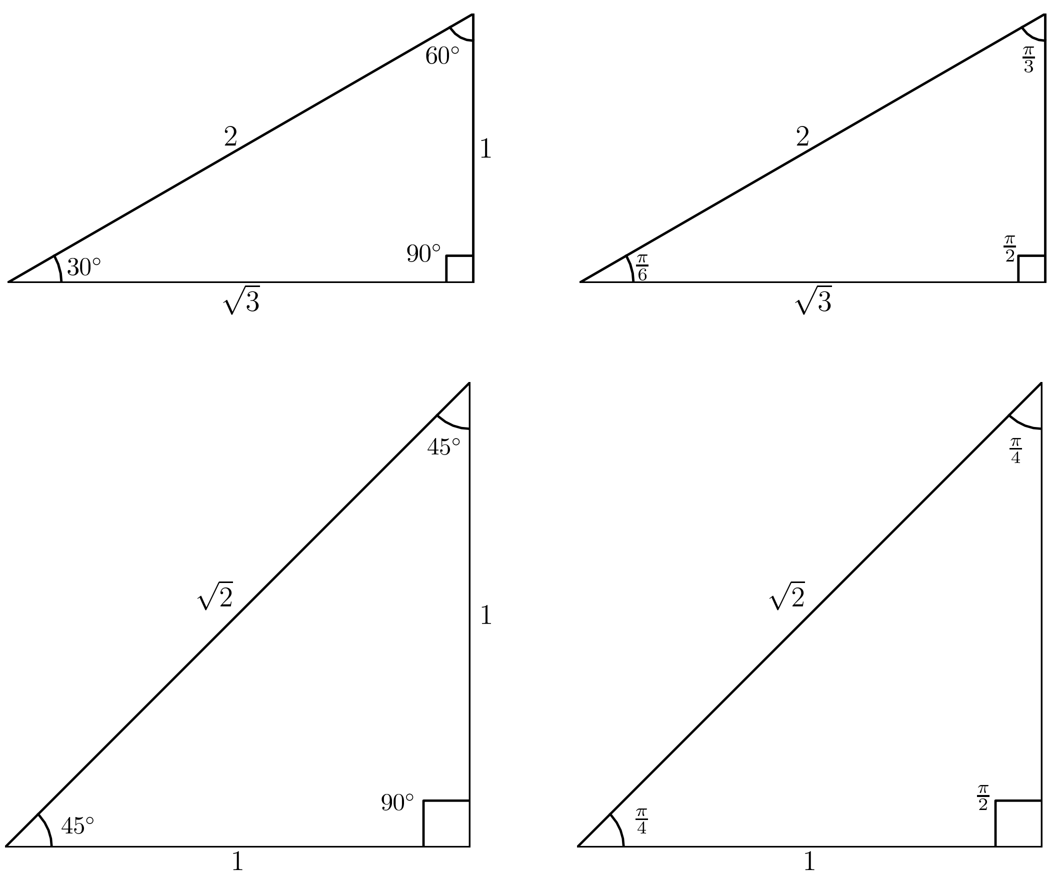
Fall 2011

We've seen how to convert the complex number $z = x + iy = re^{i\theta}$ between Cartesian coördinates (x, y) and polar coördinates (r, θ) using the transformations

$$x = r\cos\theta \quad (1a) \qquad r = \sqrt{x^2 + y^2} \quad (2a)$$

$$y = r\sin\theta \quad (1b) \qquad \theta = \mathtt{atan2}(y, x) \quad (2b)$$

together with a table of trig functions of multiples of $\pi/4$ and $\pi/6$. Rather than memorizing or reconstructing that table of trig functions, one can instead remember the following two triangles (shown with their angles labelled in degrees and then in radians):



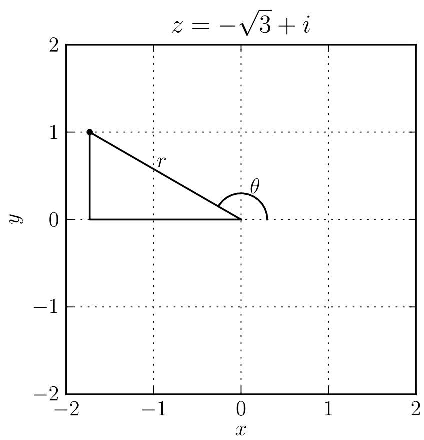
Examples

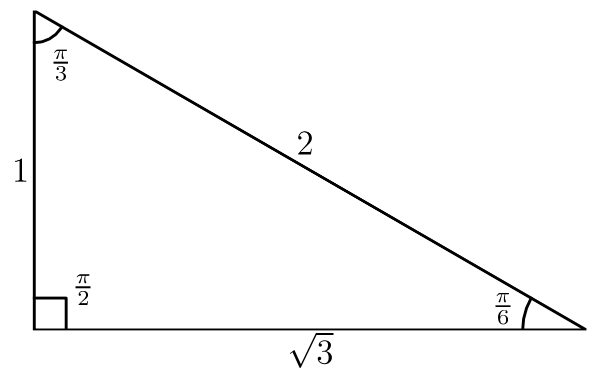
1. **Convert $z = -\sqrt{3} + i$ into polar form.**

   First we read off from $z = x + iy = -\sqrt{3} + i$ the Cartesian coördinates

$$x = -\sqrt{3} \tag{3a}$$
$$y = 1 \ . \tag{3b}$$

   This point lies in the second quadrant:





   The sides $\sqrt{3}$ and 1 fit into our 30°–60°–90° triangle, which we blow up in the right-hand side of the diagram. The modulus is $r = |z| = \sqrt{x^2 + y^2} = \sqrt{3+1} = 2$, which can also be seen from the fact that the hypotenuse of the triangle is 2. We see that the angle θ is between $\pi/2$ (90°) and π (180°). Since the small angle of the triangle is $\pi/6$, we must have

$$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6} \ , \tag{4}$$

   i.e.,

$$z = -\sqrt{3} + i = 2e^{i5\pi/6} \ . \tag{5}$$

   Note that these triangles tell us that

$$\cos\theta = \frac{x}{r} = \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} \tag{6a}$$
$$\sin\theta = \frac{y}{r} = \sin\frac{5\pi}{6} = \frac{1}{2} \ . \tag{6b}$$

   Note also that we would have got the wrong answer if we'd taken

$$\tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6} \neq \theta \ . \tag{7}$$

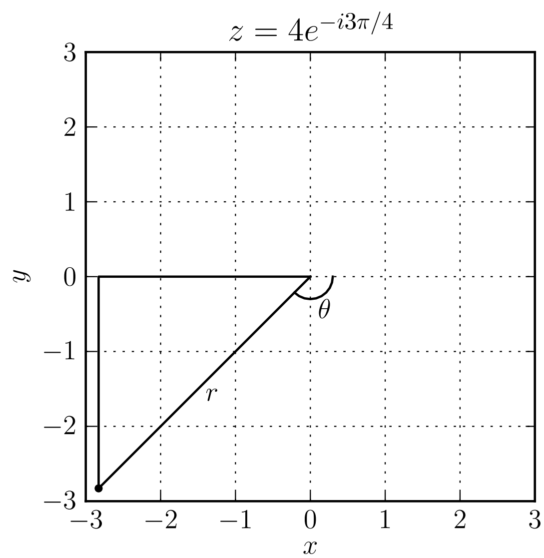
2. **Convert $z = 4e^{-i3\pi/4}$ into Cartesian form.**

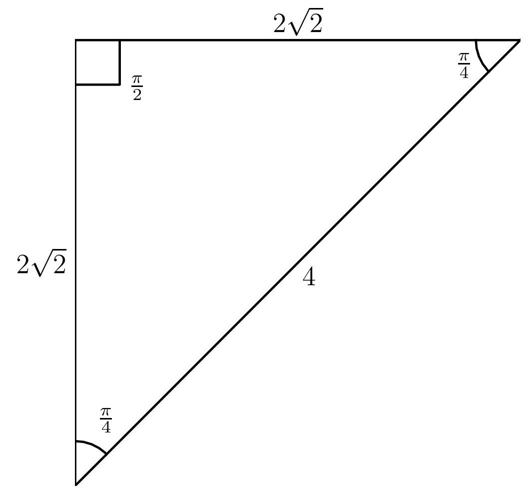
First we read off from $z = re^{i\theta} = 4e^{-i3\pi/4}$ the polar coördinates

$$r = 4 \tag{8a}$$

$$\theta = -\frac{3\pi}{4} \; . \tag{8b}$$

Since $\theta = -3\pi/4$ is between $-\pi$ ($-180°$) and $-\pi/2$ ($-90°$), the point lies in the third quadrant:





Looking at the diagram and using $\pi - 3\pi/4 = \pi/4$, we see that the angles in the triangle are 45°, which gives us the isosceles right triangle blown up on the right. We've scaled up the triangle by a factor of $2\sqrt{2}$ so that the hypotenuse is $r = 4$, from which we can read off

$$x = -2\sqrt{2} \tag{9a}$$

$$y = -2\sqrt{2} \; , \tag{9b}$$

i.e.,

$$z = 4e^{-i3\pi/4} = -2\sqrt{2} - i2\sqrt{2} \; . \tag{10}$$

Note that these triangles tell us that

$$\cos\theta = \frac{x}{r} = \cos\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} \tag{11a}$$

$$\sin\theta = \frac{y}{r} = \sin\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} \; . \tag{11b}$$