

# Physics A301: Classical Mechanics II

## Problem Set 11

Assigned 2005 April 21

Due 2005 April 28

**Show your work on all problems!** Be sure to give credit to any collaborators, or outside sources used in solving the problems.

## 1 Stability of Precession

In this problem, consider an object in which all three eigenvalues ( $I_1$ ,  $I_2$ , and  $I_3$ ) of the inertia tensor are different. Let the angular momentum vector be, initially at least, nearly parallel to the principal axis  $\hat{u}_3$ , so that  $\omega'_z \gg \omega'_x$  and  $\omega'_z \gg \omega'_y$ . Consider the evolution in the absence of torques,  $\vec{N} = \vec{0}$ .

- Write Euler's equations as expressions for  $\dot{\omega}'_x$ ,  $\dot{\omega}'_y$ , and  $\dot{\omega}'_z$ .
- Explain why, in the limit that  $\omega'^2_x + \omega'^2_y \ll \omega'^2_z$ ,  $\omega'_z$  can be treated as approximately constant.
- Given the results of part b), the first two of Euler's equations are a pair of coupled first-order differential equations for  $\omega'_x$  and  $\omega'_y$ . By appropriate differentiation and substitution, convert them into decoupled second-order equations of the form

$$\ddot{\omega}'_x = \Gamma \omega'_x \tag{1.1a}$$

$$\ddot{\omega}'_y = \Gamma \omega'_y \tag{1.1b}$$

where  $\Gamma$  is some constant made up of  $I_1$ ,  $I_2$ ,  $I_3$ , and  $\omega'_z$ .

- For the case where  $\Gamma > 0$ , define  $\beta = \sqrt{\Gamma}$  and construct the general solution to (1.1a) (which should contain two arbitrary constants). Find the corresponding  $\omega'_y$  using the results of part c).
- For the case where  $\Gamma < 0$ , define  $\Omega = \sqrt{-\Gamma}$  and construct the general solution to (1.1a) (which should contain two arbitrary constants). Find the corresponding  $\omega'_y$  using the results of part c).
- For which sign of  $\Gamma$  will  $\omega'_x$  and  $\omega'_y$  remain small compared to  $\omega'_z$ ? (This case corresponds to stable precession about the body axis  $\hat{u}_3$ , while in the other case the angle between the rotation axis and the body axis will grow for a general choice of the arbitrary constants in the general solution to the differential equation, and our approximation will break down.)
- What is the sign of  $\Gamma$ , and will we see a stable precession about  $\hat{u}_3$ , for each of the following cases?
  - If  $I_3$  is the largest eigenvalue of the inertia tensor, i.e.,  $I_3 > I_1$  and  $I_3 > I_2$ .
  - If  $I_3$  is the middle eigenvalue of the inertia tensor, i.e.,  $I_1 > I_3 > I_2$  or  $I_2 > I_3 > I_1$ .
  - If  $I_3$  is the smallest eigenvalue of the inertia tensor, i.e.,  $I_3 < I_1$  and  $I_3 < I_2$ .

## 2 Polar Wandering on the Earth

Do Problem 2 of Chapter 11 of Symon. You may find the following clarifications/tips useful:

- a) You're being asked to treat the Earth as an ellipsoid of uniform density, mass  $5.972 \times 10^{24}$  kg, and semi-axes  $a = b = 6.378 \times 10^6$  m and  $c = 6.357 \times 10^6$  m. Note that we calculated the components of the inertia tensor of a uniform-density ellipsoid in class.
- b) In order to keep the center of mass in the same place, you should put two pointlike mountains, each of mass  $5 \times 10^{-10} M_{\oplus}$ , directly opposite on the surface of the otherwise spherical Earth (of radius  $R_{\oplus} = 6.371 \times 10^6$  m).
- c) Express your answer as a fraction of the Earth's mass. Again, considering the "mountain" to be split in half and placed at antipodal points on the Earth's equator will simplify things. (Note that there's something funny about this part of the problem; it's apparently something of a trick question, but there is still a numerical answer.)

Note that we're looking for actual numbers, with appropriate units and numbers of significant figures, as the answers to the quantitative questions.

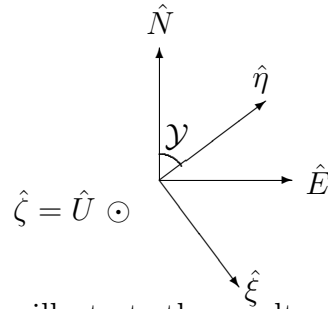
## 3 Yaw, Pitch and Roll

**Warning:** The conventions used in this problem are not standard (if such a thing even exists).

An alternative to the Euler angles we discussed in class, often used for specifying the orientation of an airplane in space, is the set of *yaw*, *pitch*, and *roll* angles. Starting with a system of unit vectors  $\hat{E}$ ,  $\hat{N}$ , and  $\hat{U}$ , pointing East, North, and Up, respectively, we can describe the orientation of an object by imagining the rotations needed to take it from an imaginary reference orientation, such as with the fuselage pointed due North and the wings pointed East and West in the horizontal plane, to its actual orientation.

1. First, we rotate the plane *clockwise* through a yaw angle  $\mathcal{Y}$  about the horizontal direction  $\hat{U}$ , which rotates the unit vectors  $\{\hat{E}, \hat{N}, \hat{U}\}$  into new unit vectors  $\{\hat{\xi}, \hat{\eta}, \hat{\zeta}\}$  (which are *not* the same as the vectors with the same names in the standard Euler angles approach). After this rotation, the plane's fuselage is pointed in the direction of  $\hat{\eta}$  and its right wing is pointed in the direction of  $\hat{\xi}$ .
2. Next, we rotate the plane *counterclockwise* through a pitch angle  $\mathcal{P}$  about the right wing direction  $\hat{\xi}$  (so that if  $0 < \mathcal{P} < \pi/2$  we are tipping the plane's nose up), which rotates the unit vectors  $\{\hat{\xi}, \hat{\eta}, \hat{\zeta}\}$  into new unit vectors  $\{\hat{X}, \hat{Y}, \hat{Z}\}$  (which have nothing to do with the coordinates of the center of mass). After this rotation, the plane's fuselage is pointed in the direction of  $\hat{Y}$  and its right wing is pointed in the direction of  $\hat{X}$ .
3. Finally, we rotate the plane *counterclockwise* through a roll angle  $\mathcal{R}$  about the fuselage direction  $\hat{Y}$ , which rotates the unit vectors  $\{\hat{X}, \hat{Y}, \hat{Z}\}$  into the final body unit vectors  $\{\hat{u}_1, \hat{u}_2, \hat{u}_3\}$ . After this rotation, the plane's fuselage is pointed in the direction of  $\hat{u}_2$  and its right wing is pointed in the direction of  $\hat{u}_1$ .

- a) The diagram below illustrates the results of the *clockwise* rotation through the yaw angle for  $0 < \mathcal{Y} < \pi/2$ . Use the diagram to write  $\hat{E}$ ,  $\hat{N}$ , and  $\hat{U}$  as linear combinations of  $\hat{\xi}$ ,  $\hat{\eta}$ , and  $\hat{\zeta}$ .



- b) Draw a similar diagram to illustrate the results of the *counterclockwise* rotation through the pitch angle for  $0 < \mathcal{P} < \pi/2$ . Use the diagram to write  $\hat{\xi}$ ,  $\hat{\eta}$ , and  $\hat{\zeta}$  as linear combinations of  $\hat{X}$ ,  $\hat{Y}$ , and  $\hat{Z}$ .
- c) Draw a similar diagram to illustrate the results of the *counterclockwise* rotation through the roll angle for  $0 < \mathcal{R} < \pi/2$ . Use the diagram to write  $\hat{X}$ ,  $\hat{Y}$ , and  $\hat{Z}$  as linear combinations of  $\hat{u}_1$ ,  $\hat{u}_2$ , and  $\hat{u}_3$ .
- d) For an object which is rotating and therefore has changing yaw, pitch, and roll angles, write the angular velocity vector  $\vec{\omega}$  as a sum of three terms involving  $\dot{\mathcal{Y}}$ ,  $\dot{\mathcal{P}}$ , and  $\dot{\mathcal{R}}$  and appropriate unit vectors. Be careful about clockwise and counter-clockwise rotations when considering the signs of the different terms.
- e) Use the results of parts a) through c) to replace the unit vectors appearing in your expression for  $\vec{\omega}$  with linear combinations of the body unit vectors  $\hat{u}_1$ ,  $\hat{u}_2$ , and  $\hat{u}_3$  and thereby find the components  $\omega'_x$ ,  $\omega'_y$ , and  $\omega'_z$  in this basis of the angular velocity vector  $\vec{\omega} = \omega'_x \hat{u}_1 + \omega'_y \hat{u}_2 + \omega'_z \hat{u}_3$ .

Note that you shouldn't have to write out any rotation matrices to do this problem.