Physics A301: Classical Mechanics II

Problem Set 9

Assigned 2005 April 7
Due 2005 April 14

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems.

1 Practice with Tensors

Consider the vectors $\vec{V} = 2\hat{y} + \hat{z}$ and $\vec{W} = 2\hat{x} - \hat{z}$. Write the following tensors i) in terms of basis tensors such as $\hat{x} \otimes \hat{x}$, $\hat{x} \otimes \hat{y}$, etc., and ii) as a matrix $\mathbf{T} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$ where

$$\overleftrightarrow{T} = T_{xx}\hat{x}\otimes\hat{x} + T_{xy}\hat{x}\otimes\hat{y} + T_{xz}\hat{x}\otimes\hat{z} + T_{yx}\hat{y}\otimes\hat{x} + T_{yy}\hat{y}\otimes\hat{y} + T_{yz}\hat{y}\otimes\hat{z} + T_{zx}\hat{z}\otimes\hat{x} + T_{zy}\hat{z}\otimes\hat{y} + T_{zz}\hat{z}\otimes\hat{z}$$

(As an example, if $\vec{a} = 2\hat{y}$ and $\vec{b} = 3\hat{x}$, $\overleftrightarrow{Q} = \vec{a} \otimes \vec{b} = 6(\hat{y} \otimes \hat{x})$, which means the only non-zero component of $\overleftrightarrow{Q}$ is $Q_{yx} = 6$, and thus $\mathbf{Q} = \begin{pmatrix} 0 & 0 & 0 \\ 6 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.)

a) $\overleftrightarrow{A} = \vec{V} \otimes \vec{V}$; b) $\overleftrightarrow{B} = \vec{W} \otimes \vec{W}$; c) $\overleftrightarrow{C} = \vec{V} \otimes \vec{W}$;

d) $\overleftrightarrow{D} = \vec{W} \otimes \vec{V}$; e) $\overleftrightarrow{E} = \frac{\overleftrightarrow{C} + \overleftrightarrow{D}}{2}$; f) $\overleftrightarrow{F} = \frac{\overleftrightarrow{C} - \overleftrightarrow{D}}{2}$;

2 Tensors and Rotating Coördinates

a) Show that the centrifugal force can be written as a tensor dotted into the position vector:

$$-m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \overleftrightarrow{M} \cdot \vec{r}$$

First write the tensor $\overleftrightarrow{M}$ in terms of tensor products of vectors, along with the unit tensor $\overleftrightarrow{1}$, then write its components in terms of the (Cartesian) components of $\vec{\omega}$.

b) Consider two sets of Cartesian basis vectors: one fixed set $\{\hat{x}, \hat{y}, \hat{z}\} = \{\hat{e}_i | i = 1, 2, 3\}$ and one set $\{\hat{x}', \hat{y}', \hat{z}'\} = \{\hat{e}'_i | i = 1, 2, 3\}$ rotating with instantaneous angular velocity ω. (When studying rotating coördinates in Chaper 7, we called these $\{\hat{x}^*, \hat{y}^*, \hat{z}^*\}$.) The components of the tensor in these two bases are as usual[1]

$$\overleftrightarrow{T} = \sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}(\hat{e}_i \otimes \hat{e}_j) = \sum_{i=1}^{3}\sum_{j=1}^{3} T'_{ij}(\hat{e}'_i \otimes \hat{e}'_j) \ . \tag{2.1}$$

[1] We should probably call the primed components something like $\{T_{i'j'}\}$ or $\{T_{ij'}\}$ but with two indices it gets kind of cumbersome, so we fall back on the rationalization that $\{T'_{ij}\}$ are the elements of the matrix $\mathbf{T}'$.

Define “primed” and “unprimed” time derivatives

$$\frac{d\overleftrightarrow{T}}{dt} = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{dT_{ij}}{dt}(\hat{e}_i \otimes \hat{e}_j) \tag{2.2a}$$

$$\frac{d'\overleftrightarrow{T}}{dt} = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{dT'_{ij}}{dt}(\hat{e}'_i \otimes \hat{e}'_j) \tag{2.2b}$$

and show that

$$\frac{d\overleftrightarrow{T}}{dt} = \frac{d'\overleftrightarrow{T}}{dt} + \vec{\omega} \times \overleftrightarrow{T} - \overleftrightarrow{T} \times \vec{\omega} \tag{2.3}$$

where the cross product of a vector with a tensor is defined in the obvious way, starting from the dyads:

$$(\vec{A} \otimes \vec{B}) \times \vec{V} = \vec{A} \otimes (\vec{B} \times \vec{V}) \tag{2.4a}$$

$$\vec{V} \times (\vec{A} \otimes \vec{B}) = (\vec{V} \times \vec{A}) \otimes \vec{B} \tag{2.4b}$$

3 Kinetic Energy

Consider a distribution of N point particles all rotating with angular velocity $\vec{\omega}$ relative to a fixed origin.

a) Write the velocity $\dot{\vec{r}}_k$ of the kth particle in terms of the angular velocity $\vec{\omega}$ and its position vector $\vec{r}_k$.

b) Write the total kinetic energy

$$T = \sum_{k=1}^{N} \frac{1}{2} m_k (\dot{\vec{r}}_k \cdot \dot{\vec{r}}_k)$$

in terms of $\vec{\omega}$ and the positions of the particles.

c) Prove the vector identity

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C}) \tag{3.1}$$

by applying the identities

$$\vec{U} \cdot (\vec{V} \times \vec{W}) = (\vec{U} \times \vec{V}) \cdot \vec{W} \tag{3.2}$$

and

$$\vec{U} \times (\vec{V} \times \vec{W}) = \vec{V}(\vec{U} \cdot \vec{W}) - \vec{W}(\vec{U} \cdot \vec{V}) \tag{3.3}$$

with suitable choices for $\vec{U}$, $\vec{V}$, and $\vec{W}$ each time.

d) Use the identity (3.1) to rewrite T in a form containing no cross products.

e) Use tensor notation to pull $\vec{\omega}$ outside the sum as we did in the derivation of $\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}$, and write the kinetic energy in the form

$$T = \vec{\omega} \cdot \overleftrightarrow{A} \cdot \vec{\omega} \tag{3.4}$$

writing the tensor $\overleftrightarrow{A}$ in terms of the masses and position vectors of the particles.

f) How is $\overleftrightarrow{A}$ related to the inertia tensor $\overleftrightarrow{I}$?