Physics A301: Classical Mechanics II

Problem Set 8

Assigned 2005 March 17
Due 2005 March 31

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems.

1 Partial Derivatives of the Hamiltonian

Note: when taking partial derivatives of the Hamiltonian, we usually consider it to be a function of coördinates and momenta rather than of velocities. In this problem, we will explicitly consider H as a function of different sets of arguments and compare the partial derivatives with different quantities held constant.

Consider the Lagrangian

$$L(q,\dot{q},t) = \frac{aq\dot{q}^2}{2} + b\dot{q}\sin\omega t - \frac{kq^2}{2}$$

where a, b, ω and k are all constants included to get the dimensions right.

a) Take the partial derivatives $\left(\frac{\partial L}{\partial q}\right)_{\dot{q},t}$, $\left(\frac{\partial L}{\partial \dot{q}}\right)_{q,t}$, and $\left(\frac{\partial L}{\partial t}\right)_{q,\dot{q}}$.

b) Find the conjugate momentum $p(q,\dot{q},t) = \left(\frac{\partial L}{\partial \dot{q}}\right)_{q,t}$.

c) Invert the results of part b) to obtain $\dot{q}(q,p,t)$.

d) Construct the Hamiltonian $H(q,\dot{q},t) = p(q,\dot{q},t)\,\dot{q} - L(q,\dot{q},t)$, writing it first as a function of the coördinate and velocity with no reference to the momentum. (This is not how we usually do it, but we're trying to prove a point here.)

e) Take the partial derivatives $\left(\frac{\partial H}{\partial q}\right)_{\dot{q},t}$, $\left(\frac{\partial H}{\partial \dot{q}}\right)_{q,t}$, and $\left(\frac{\partial H}{\partial t}\right)_{q,\dot{q}}$. Show that $\left(\frac{\partial H}{\partial q}\right)_{\dot{q},t} \neq -\left(\frac{\partial L}{\partial q}\right)_{\dot{q},t}$ and $\left(\frac{\partial H}{\partial t}\right)_{q,\dot{q}} \neq -\left(\frac{\partial L}{\partial t}\right)_{q,\dot{q}}$.

f) Use the results of parts c) and d) to rewrite the Hamiltonian as a function $H(q,p,t)$ of the coördinate and momentum with no reference to the velocity.

g) Take the partial derivatives $\left(\frac{\partial H}{\partial q}\right)_{p,t}$, $\left(\frac{\partial H}{\partial p}\right)_{q,t}$, and $\left(\frac{\partial H}{\partial t}\right)_{q,p}$. These will be functions of q, p, and t.

h) Use the results of part b) to write all three partial derivatives from part g) as functions of q, $\dot{q}$, and t, and show that $\left(\frac{\partial H}{\partial q}\right)_{p,t} = -\left(\frac{\partial L}{\partial q}\right)_{\dot{q},t}$, and $\left(\frac{\partial H}{\partial t}\right)_{q,p} = -\left(\frac{\partial L}{\partial t}\right)_{q,\dot{q}}$.

2 Two-Body Problem Revisited

Consider the Lagrangian

$$L = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{Z}^2}{2} + \frac{\mu\dot{r}^2}{2} + \frac{\mu r^2\dot{\theta}^2}{2} + \frac{\mu r^2 \sin^2\theta\,\dot{\phi}^2}{2} + \frac{GM\mu}{r} - MgZ$$

which you found in problem 2 on problem set 5.

a) Construct the six conjugate momenta p_X, p_Y, p_Z, p_r, p_θ, and p_ϕ as functions of the coördinates $\{X, Y, Z, r, \theta, \phi\}$ and velocities $\{\dot{X}, \dot{Y}, \dot{Z}, \dot{r}, \dot{\theta}, \dot{\phi}\}$.

b) Invert those relationships to find the six generalized velocities $\dot{X}$, $\dot{Y}$, $\dot{Z}$, $\dot{r}$, $\dot{\theta}$, and $\dot{\phi}$ in terms of the coördinates and momenta.

c) Construct the Hamiltonian as a function of the coördinates and momenta with no reference to any of the velocities in your final result.

d) Write all twelve of Hamilton's equations. Which coördinates are ignorable?

3 Principle of Least Action

Consider a family of curves $x_\alpha(t) = x(t) + \alpha\xi(t)$, where $\xi(t)$ is an otherwise arbitrary function which vanishes at times t_i and t_f [i.e., $\xi(t_i) = 0 = \xi(t_f)$].

a) Calculate the derivatives $\frac{\partial x_\alpha}{\partial \alpha}$ and $\frac{\partial \dot{x}_\alpha}{\partial \alpha}$ where $\dot{x}_\alpha$ is the time derivative of $x_\alpha(t)$ (implicitly at constant α, since α is a single number and not a function of time).

b) Consider a function $L(x, \dot{x}, t)$, from which we can derive a function $L_\alpha(t) = L(x_\alpha(t), \dot{x}_\alpha(t), t)$. Use the chain rule to write $\frac{\partial L_\alpha}{\partial \alpha}$ in terms of the partial derivatives $\left.\frac{\partial L}{\partial x}\right|_{x=x_\alpha}$ and $\left.\frac{\partial L}{\partial \dot{x}}\right|_{x=x_\alpha}$.

c) Define the function

$$S(\alpha) = \int_{t_i}^{t_f} L_\alpha(t)\,dt$$

and use the results of the previous two parts to write $S'(\alpha)$ as an integral containing ξ, $\dot{\xi}$, $\left.\frac{\partial L}{\partial x}\right|_{x=x_\alpha}$ and $\left.\frac{\partial L}{\partial \dot{x}}\right|_{x=x_\alpha}$.

d) Use integration by parts (i.e., $\int_{t_i}^{t_f} x\frac{dy}{dt}\,dt = xy\big|_{t_i}^{t_f} - \int_{t_i}^{t_f} y\frac{dx}{dt}\,dt$) to convert the term involving $\dot{\xi}$ into a term involving ξ.

e) Show that if $x(t)$ satisfies the Lagrange equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$, then $S(\alpha)$ has a local extremum at $\alpha = 0$.

S is called the action, and Lagrange's equations are equivalent to the condition that the action be smaller for the classical trajectory than for any "nearby" trajectory.