Physics A301: Classical Mechanics II

Problem Set 7

Assigned 2005 March 10
Due 2005 March 17

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems.

1 Conserved Quantities

Consider two point-like balls each of mass m connected by a spring of spring constant k and unstretched length ℓ_0, free to move in *two* dimensions. There is no gravity in this problem.

a) Construct the Lagrangian $L(\ell, \phi, X, Y, \dot{\ell}, \dot{\phi}, \dot{X}, \dot{Y}, t)$ for this system, using as your generalized coördinates the actual length $\ell(t)$ of the spring, the angle $\phi(t)$ it makes with the horizontal, and the Cartesian coördinates $X(t)$ and $Y(t)$ of the center of mass. (Hint: the kinetic energy is most easily found by using the result from the two-body problem–e.g., problem 2b)i) on problem set 5–and writing the relative coördinates as $x = \ell\cos\phi$ and $y = \ell\sin\phi$.)

b) Calculate the partial derivatives $\frac{\partial L}{\partial \ell}$, $\frac{\partial L}{\partial \phi}$, $\frac{\partial L}{\partial X}$, $\frac{\partial L}{\partial Y}$, and $\frac{\partial L}{\partial t}$.

c) Based on the results of part b), construct *four* independent functions of the generalized coördinates (ℓ, ϕ, X, Y) and generalized velocities $(\dot{\ell}, \dot{\phi}, \dot{X}, \dot{Y})$, and t which are constants of the motion. (You shouldn't need to write out the Lagrange equations to do this, just use the symmetries of the Lagrangian.) State what each of them is physically.

d) Suppose the constant values of the four expessions in part c) are known; solve algebraically for each of the four generalized velocities as functions of the generalized coördinates and the values of the four constants of the motion.

2 Electromagnetic Lagrangian

Consider the Lagrangian

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) = \frac{1}{2} m\, \dot{\vec{r}} \cdot \dot{\vec{r}} - Q\varphi(\vec{r}, t) + Q\, \dot{\vec{r}} \cdot \vec{A}(\vec{r}, t)$$

which describes a charged particle of mass m and charge Q moving in an external electromagnetic field derived from a scalar potential $\varphi(\vec{r}, t)$ and vector potential $\vec{A}(\vec{r}, t)$.

a) Work out the Lagrange equations $\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0$ and $\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0$, and verify that they are equivalent to the y and z components of the Lorentz force law

$$m\ddot{\vec{r}} = Q(\vec{E} + \dot{\vec{r}} \times \vec{B}) \tag{2.1}$$

where the electromagnetic fields are derived as usual from the potentials by

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t} \tag{2.2a}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{2.2b}$$

[we showed in class that $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$ is equivalent to the x component of (2.1)].

b) Work out the conjugate momenta $p_x = \frac{\partial L}{\partial \dot{x}}$ etc. What conditions on $\varphi(\vec{r}, t)$ and $\vec{A}(\vec{r}, t)$ are needed for each of these to be a constant of the motion?

c) Work out the Hamiltonian $H = p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - L$ as a function of $\vec{r}$, $\dot{\vec{r}}$ and t. What conditions on $\varphi(\vec{r}, t)$ and $\vec{A}(\vec{r}, t)$ are needed for it to be a constant of the motion?

3 Time-Dependent Lagrangian

Consider the Lagrangian

$$L(\theta, \dot{\theta}, t) = \frac{1}{2}m\ell^2\dot{\theta}^2 + m(v_0 + at)\ell\dot{\theta}\cos\theta + \frac{1}{2}m(v_0 + at)^2 + mg\ell\cos\theta$$

a) Calculate the partial derivatives $\frac{\partial L}{\partial \theta}$, $\frac{\partial L}{\partial \dot{\theta}}$, and $\frac{\partial L}{\partial t}$.

b) Calculate the total derivative

$$\frac{d}{dt}L(\theta(t), \dot{\theta}(t), t) = \frac{\partial L}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial L}{\partial \dot{\theta}}\frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial t}$$

c) Use the equation of motion

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta - \frac{a}{\ell}\cos\theta$$

to replace $\ddot{\theta}$ in your expression for the total derivative $\frac{dL}{dt}$ and demonstrate that $\frac{dL}{dt} \neq \frac{\partial L}{\partial t}$

d) Construct the Hamiltonian

$$H = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L$$

as a function of θ, $\dot{\theta}$, and t.

e) Using the kinetic and potential energies

$$T = \frac{1}{2}m\ell^2\dot{\theta}^2 + m(v_0 + at)\ell\dot{\theta}\cos\theta + \frac{1}{2}m(v_0 + at)^2$$

$$V = -mg\ell\cos\theta$$

construct the total energy $E = T + V$, and calculate $E - H$.