Physics A301: Classical Mechanics II

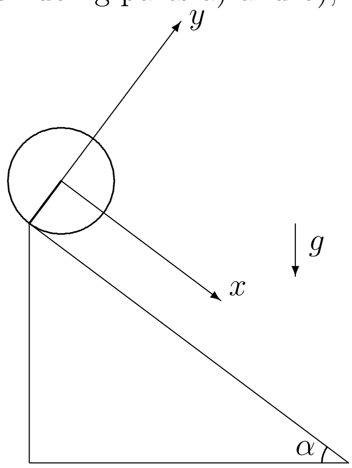
Problem Set 6

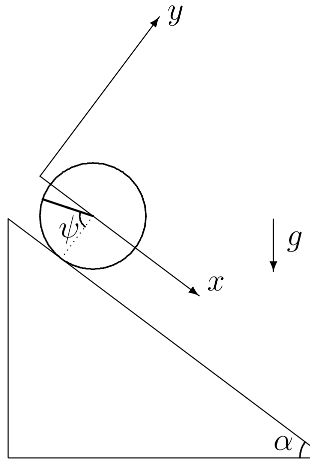
Assigned 2005 March 3
Due 2005 March 10

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems.

1 Rolling Down an Incline

Consider a solid cylinder of mass M, constant density, and radius R rolling down a plane which is inclined at an angle α to the horizontal. Define as generalized coördinates the Cartesian coördinates x and y of the cylinder's center of mass and the angle ψ through which the cylinder has rotated. Let the x axis be parallel to and the y axis be perpendicular to the inclined plane, and let the origins of coördinates be defined so that x, y, and ψ are all zero when the cylinder is at the top of the incline. When doing parts a) and b), do *not* impose the constraints described in part c).





a) Write the total kinetic energy $T(x, y, \psi, \dot{x}, \dot{y}, \dot{\psi})$ of the cylinder, including both translational and rotational kinetic energy. [See section 5.2 of Symon for the form of rotational kinetic energy, and Symon's equation (5.90) for the calculation of the moment of inertia of a disc or cylinder.]

b) Find the gravitational potential energy $V(x, y, \psi)$ of the cylinder if there is a constant downward gravitational field of magnitude g and the zero of the potential energy is chosen to be when the cylinder is at the top of the incline.

c) Write the equations of constraint which describe the fact that

   i) the cylinder does not leave the surface of the incline

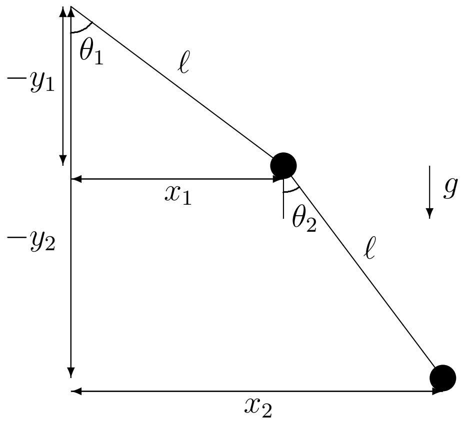
   ii) the cylinder rolls without slipping

d) Construct the modified Lagrangian $L(x, y, \psi, \lambda_1, \lambda_2, \dot{x}, \dot{y}, \dot{\psi})$ including two Lagrange multipliers to enforce the two constraints.

e) What is the physical nature of the constraining force associated with each Lagrange multiplier? (E.g., tension, normal force, etc.)

2 The Double Pendulum

Consider the system illustrated in the figure below: a mass m hangs from a fixed suspension point by a rod of fixed length ℓ, and a second mass, also of mass m, hangs from the first mass by another rod, also of fixed length ℓ.



a) Define a coördinate system in which the origin is at the suspension point of the first pendulum, the x direction is to the right, and the y direction is up. Let the position the first mass be given by Cartesian coördinates $(x_1(t), y_1(t))$ and the second by Cartesian coördinates $(x_2(t), y_2(t))$. Write the kinetic energies T_1 and T_2 and potential energies V_1 and V_2 in terms of the Cartesian coördinates and velocities.

b) If the first rod makes an angle $\theta_1(t)$ with the vertical and the second rod makes an angle $\theta_2(t)$ with the vertical, write

   i) $x_1(t)$ and $y_1(t)$ in terms of ℓ and $\theta_1(t)$;

   ii) $x_2(t)$ and $y_2(t)$ in terms of ℓ, $\theta_2(t)$, $x_1(t)$, and $y_1(t)$;

   iii) $x_2(t)$ and $y_2(t)$ in terms of ℓ, $\theta_2(t)$, and $\theta_1(t)$.

   iv) $\dot{x}_1(t)$, $\dot{y}_1(t)$, $\dot{x}_2(t)$, and $\dot{y}_2(t)$ in terms of ℓ, $\theta_1(t)$, $\theta_2(t)$, $\dot{\theta}_1(t)$, and $\dot{\theta}_2(t)$,

c) Write the kinetic energies T_1 and T_2 and potential energies V_1 and V_2 in terms of the parameters ℓ and m, the angles $\theta_1(t)$ and $\theta_2(t)$, and their derivatives $\dot{\theta}_1(t)$ and $\dot{\theta}_2(t)$.

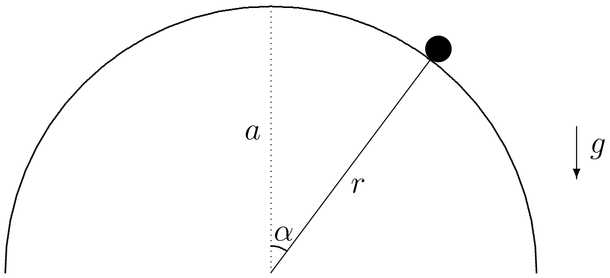
d) Write the Lagrangian $L(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, t)$ describing the system, with θ_1 and θ_2 taken as the generalized coördinates.

e) Use the Lagrange equations to find the equations of motion satisfied by $\theta_1(t)$ and $\theta_2(t)$.

You should not assume either angle is small at any point in this problem.

3 Sliding Off the Sphere

Consider the situation illustrated in the figure below. A point particle of mass m slides without friction on the surface of a sphere of radius a. If it is pushed from the top of the sphere with negligible initial speed, it will slide part of the way along the sphere and then lose contact with the surface. In this problem you will work out how far down the sphere this occurs.



a) Using as the generalized coördinates the quantities r (the distance from the center of the sphere) and α (the angle down from the north pole of the sphere), write the kinetic energy $T(r, \alpha, \dot{r}, \dot{\alpha})$ of the particle. (Do not assume in this part or the next that the particle stays on the surface of the sphere; that is a constraint which will be imposed later.)

b) If the sphere is in a constant downward gravitational field of magnitude g, write the potential energy $V(r, \alpha)$ of the particle, setting the zero of the potential energy to be in the equatorial plane of the sphere.

c) Write the equation of constraint $h(r, \alpha) = 0$ that says that the particle remains on the surface of the sphere.

d) Construct the modified Lagrangian $\tilde{L}(r, \alpha, \dot{r}, \dot{\alpha}, \lambda) = T(r, \alpha, \dot{r}, \dot{\alpha}) - V(r, \alpha) + \lambda h(r, \alpha)$

e) Take the appropriate derivatives to form the three Lagrange equations. (Do not solve any of them yet, and in particular, don't impose the constraint in this step.)

f) Fix the values of r, $\dot{r}$, and $\ddot{r}$ by imposing the constraint, and solve one of the remaining two equations for λ as a function of α and $\dot{\alpha}$ (along with the constants m, g, and a).

g) Rather than trying to work with the remaining Lagrange equation, it's easier to get a relationship between $\dot{\alpha}$ and α from conservation of energy. Use the fact that $T + V$ is a constant and the initial condition that the particle's speed is negligible at the top of the sphere to solve for $\dot{\alpha}$ as a function of α (along with the constants m, g, and a).

h) Substitute the result of part g) into part f) to obtain λ as a function of α (along with the constants m, g, and a).

i) The constraining force in this problem is the normal force, which can only point outwards. Thus the particle leaves the sphere when the Lagrange multiplier λ (which is proportional to the normal force) changes sign. Find the angle α at which this occurs.