Physics A301: Classical Mechanics II

Problem Set 5

Assigned 2005 February 24
Due 2005 March 3

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems.

1 Lagrangian in Rotating Coördinates

Consider a particle moving in three dimensions under the influence of a potential $V(\vec{x})$. Let x, y, and z be ordinary inertial Cartesian coördinates. Let x^*, y^*, and z^* be a set of Cartesian coördinates whose axes are rotating with respect to the unstarred axes with a constant angular velocity $\vec{\omega} = \omega\hat{z}$, so that the transformation between the two coördinate systems is given by

$$x = x^* \cos\omega t - y^* \sin\omega t \tag{1.1a}$$
$$y = x^* \sin\omega t + y^* \cos\omega t \tag{1.1b}$$
$$z = z^* \tag{1.1c}$$

a) Find the velocities $\dot{x}$, $\dot{y}$, and $\dot{z}$ in terms of the generalized velocities $\dot{x}^*$, $\dot{y}^*$, and $\dot{z}^*$, the generalized coördinates x^*, y^*, and z^*, and time t.

b) Write the kinetic energy T as a function of the generalized coördinates and velocities.

c) Construct the Lagrangian L, written as a function of the generalized coördinates and velocities. (This will contain the unspecified potential $V(x^*, y^*, z^*, t)$.)

d) Calculate the partial derivatives $\frac{\partial L}{\partial x^*}$, $\frac{\partial L}{\partial y^*}$, and $\frac{\partial L}{\partial z^*}$.

e) Calculate the partial derivatives $\frac{\partial L}{\partial \dot{x}^*}$, $\frac{\partial L}{\partial \dot{y}^*}$, and $\frac{\partial L}{\partial \dot{z}^*}$.

f) Calculate the time derivatives $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^*}$, $\frac{d}{dt}\frac{\partial L}{\partial \dot{y}^*}$, and $\frac{d}{dt}\frac{\partial L}{\partial \dot{z}^*}$.

g) Construct all three of Lagrange's equations and write them as equations for $\ddot{x}^*$, $\ddot{y}^*$, and $\ddot{z}^*$.

h) Work out the equations of motion in the starred coördinate system independently using the formalism developed in Chapter Seven for rotating coördinate systems, and verify that you get the same equations for $\ddot{x}^*$, $\ddot{y}^*$, and $\ddot{z}^*$.

2 Two-Body Problem from Lagrangian Standpoint

Consider two particles, of mass m_1 and m_2, with position vectors $\vec{r}_1$ and $\vec{r}_2$, moving in a constant external gravitational field $\vec{g} = -g\hat{z}$ and attracting each other gravitationally, so that the potential energy of the system is

$$V(\vec{r}_1, \vec{r}_2) = -\frac{Gm_1m_2}{|\vec{r}_1 - \vec{r}_2|} + m_1gz_1 + m_2gz_2 \tag{2.1}$$

a) Verify that (2.1) gives the correct forces on particles 1 and 2 by calculating the gradients with respect to $\vec{r}_1$ and $\vec{r}_2$.

b) Define the relative position vector and the center-of-mass vector

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \tag{2.2a}$$

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \tag{2.2b}$$

   i) Convert the kinetic energy $T = \frac{1}{2}m_1\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_1 + \frac{1}{2}m_2\dot{\vec{r}}_2 \cdot \dot{\vec{r}}_2$ into a function of $\dot{\vec{r}}$ and $\dot{\vec{R}}$. Simplify this function by defining $M = m_1 + m_2$ and $\mu = \frac{m_1m_2}{m_1+m_2}$, so that your final version contains M and μ, but not m_1 or m_2.

   ii) Convert the potential energy (2.1) into a function of $\vec{r}$ and $\vec{R}$, again making the appropriate substitutions so that it contains M and μ, instead of m_1 or m_2.

c) Consider a set of generalized coördinates which consist of X, Y, and Z (the Cartesian components of $\vec{R}$) along with r, θ, and ϕ (the description of the "position" vector $\vec{r}$ in spherical coördinates).

   i) Construct the Lagrangian in the generalized coördinates.

   ii) Calculate the appropriate derivatives and work out all six of Lagrange's equations.

3 Lagrangian in Parabolic Coördinates

Symon Chapter Nine, Problem Three.

Note that the "momenta" Symon refers to are $p_f = \frac{\partial L}{\partial \dot{f}}$ and $p_h = \frac{\partial L}{\partial \dot{h}}$ (there's a typo in my edition where Symon says Ph when he means p_h) and are *not* components of the momentum vector $\vec{p}$.