Physics A300: Classical Mechanics I

Problem Set 9

Assigned 2004 November 23
Due 2004 December 2

1 Conic Sections (Kepler's First Law)

Demonstrate that the orbit

$$r(1 + \varepsilon \cos\phi) = \alpha \tag{1.1}$$

with constants $\alpha > 0$ and $\varepsilon \geq 0$ is indeed a conic section with eccentricity ε, semimajor axis $\alpha/(1-\varepsilon^2)$, and one focus at $r = 0$ as follows:

a) Consider the points $\mathcal{P} \equiv (x, y)$, $\mathcal{O} \equiv (0, 0)$, $\mathcal{F}_\pm \equiv (\pm 2c, 0)$, (where $c > 0$) and the line $\mathcal{L} \equiv x = 2p > 0$. Calculate the following distances in Cartesian coördinates, then convert your results into the standard polar coördinates using $x = r\cos\phi$ and $y = r\sin\phi$, simplifying as much as possible. (It may be useful to sketch these objects in the x-y plane.)

   i) the length $d_{\mathcal{OP}}$ of the straight line segment from $\mathcal{O}$ to $\mathcal{P}$

   ii) the length $d_{\mathcal{F}_\pm\mathcal{P}}$ of the straight line segment from $\mathcal{F}_\pm$ to $\mathcal{P}$

   iii) the distance $d_{\mathcal{LP}}$ between the point $\mathcal{P}$ and the line $\mathcal{L}$

b) A circle of radius a centered at $\mathcal{O}$ is the set of all points a distance a from $\mathcal{O}$:

$$d_{\mathcal{OP}} = a \tag{1.2}$$

Show that when $\varepsilon = 0$, (1.1) is equivalent to (1.2) for a suitable choice of a, and find this a in terms of α.

c) An ellipse of semimajor axis $a > 0$ with foci at $\mathcal{F}_-$ and $\mathcal{O}$ is the set of all points such that the sum of their distances from the two foci is $2a$:

$$d_{\mathcal{F}_-\mathcal{P}} + d_{\mathcal{OP}} = 2a \tag{1.3}$$

Show that when $0 < \varepsilon < 1$, (1.1) is equivalent to (1.3) for a suitable choice of a and c, and find these values in terms of α and ε. (Hint: this is easiest if you solve (1.3) for $d_{\mathcal{F}_-\mathcal{P}}$, square it, and set it equal to the square of the result from part a)ii), using (1.1) to eliminate $\cos\phi$, and requiring equality for any value of r.)

d) A parabola with focus $\mathcal{O}$ and directrix $\mathcal{L}$ is the set of all points equidistant from $\mathcal{O}$ and $\mathcal{L}$:

$$d_{\mathcal{LP}} = d_{\mathcal{OP}} \tag{1.4}$$

Show that when $\varepsilon = 1$, (1.1) is equivalent to (1.4) for a suitable choice of p, and find this p in terms of α.

e) The left branch of a hyperbola of semimajor axis $a < 0$ with foci at $\mathcal{O}$ and $\mathcal{F}_+$ is the set of all points such that the difference of their distances from the two foci is $-2a > 0$:

$$d_{\mathcal{F}_+\mathcal{P}} - d_{\mathcal{O}\mathcal{P}} = -2a \tag{1.5}$$

Show that when $\varepsilon > 1$, (1.1) is equivalent to (1.5) for a suitable choice of a and c, and find these values in terms of α and ε. (Hint: this is easiest if you solve (1.5) for $d_{\mathcal{F}_+\mathcal{P}}$, square it, and set it equal to the square of the result from part a)ii), using (1.1) to eliminate $\cos\phi$, and requiring equality for any value of r.)

2 Cartesian Form of Ellipse (Kepler's Third Law–sort of)

The demonstration of Kepler's third law in section 3.15 of Symon rests on the fact that the area of an ellipse is πab, which essentially comes down to the fact that an ellipse is the shape you get when you stretch a circle by different amounts in perpendicular directions. This in turn is apparent from the standard equation for an ellipse of semiäxes a and b centered at the point (x_c, y_c):

$$\frac{(x - x_c)^2}{a^2} + \frac{(y - y_c)^2}{b^2} = 1 \tag{2.1}$$

Show that this is indeed satisfied, with $(x_c, y_c) = (-a\varepsilon, 0)$, for any point on the curve

$$r = \frac{a(1 - \varepsilon^2)}{1 + \varepsilon\cos\phi} \tag{2.2}$$

where $0 \le \varepsilon < 1$, and the semiminor axis is given by $b = a\sqrt{1 - \varepsilon^2}$.

3 Properties of a Mass Distribution

Consider four identical particles, each of mass m, each moving in counter-clockwise around a circle of radius a in the x-y plane, centered at the origin, at constant angular velocity $\Omega > 0$, with their positions evenly spaced around the circle.

a) Sketch this situation, and label the particles 1 through 4.

b) Write the position vectors $\vec{r}_1(t)$, $\vec{r}_2(t)$, $\vec{r}_3(t)$, and $\vec{r}_4(t)$ if particle 1 crosses the positive x axis at $t = 0$. (Assume the orbital plane is $z = 0$.)

c) Calculate the velocities $\dot{\vec{r}}_1(t)$, $\dot{\vec{r}}_2(t)$, $\dot{\vec{r}}_3(t)$, and $\dot{\vec{r}}_4(t)$.

d) Calculate *explicitly*

   i) the total mass M;
   ii) the total momentum $\vec{P}$;
   iii) the position vector $\vec{R}$ of the center of mass;
   iv) the total angular momentum $\vec{L}$;
   v) the total kinetic energy T

   You don't need to calculate any components of vectors which vanish as a result of the motion being confined to a plane, but you should calculate all other components of the relevant vectors, even if they turn out to be zero due to symmetry.