Physics A300: Classical Mechanics I

Problem Set 7

Assigned 2004 November 2
Due 2004 November 9

Show your work on all problems!

1 Spherical Coördinates

Consider the unit vectors

$$\hat{r} = \sin\theta\cos\phi\ \hat{x} + \sin\theta\sin\phi\ \hat{y} + \cos\theta\ \hat{z} \tag{1.1a}$$

$$\hat{\theta} = \cos\theta\cos\phi\ \hat{x} + \cos\theta\sin\phi\ \hat{y} - \sin\theta\ \hat{z} \tag{1.1b}$$

$$\hat{\phi} = -\sin\phi\ \hat{x} + \cos\phi\ \hat{y} \tag{1.1c}$$

a) Using the usual expression for the dot product in terms of Cartesian components [e.g., Symon's Eq. (3.23)], calculate explicitly the six independent inner products $\hat{r}\cdot\hat{r}$, $\hat{r}\cdot\hat{\theta}$, $\hat{r}\cdot\hat{\phi}$, $\hat{\theta}\cdot\hat{\theta}$, $\hat{\theta}\cdot\hat{\phi}$ and $\hat{\phi}\cdot\hat{\phi}$, and thereby show that the unit vectors defined in (1.1) are themselves an orthonormal basis.

b) Using the usual expression for the dot product in terms of Cartesian components [e.g., Symon's Eq. (3.33)], calculate $\hat{r}\times\hat{\theta}$, $\hat{\theta}\times\hat{\phi}$, and $\hat{\phi}\times\hat{r}$.

c) By differentiating the form (1.1), calculate the nine partial derivatives $\frac{\partial\hat{r}}{\partial r}$, $\frac{\partial\hat{r}}{\partial\theta}$, $\frac{\partial\hat{r}}{\partial\phi}$, $\frac{\partial\hat{\theta}}{\partial r}$, $\frac{\partial\hat{\theta}}{\partial\theta}$, $\frac{\partial\hat{\theta}}{\partial\phi}$, $\frac{\partial\hat{\phi}}{\partial r}$, $\frac{\partial\hat{\phi}}{\partial\theta}$ and $\frac{\partial\hat{\phi}}{\partial\phi}$. First express your results in terms of the Cartesian basis vectors (with components written in terms of the spherical coördinates r, θ, and ϕ). Then use your results along with (1.1) to verify Symon's Eq. (3.99) for the derivatives written purely in terms of the spherical coördinates and the corresponding basis.

2 The Curl

a) If $a(\vec{r})$ is a scalar field and $\vec{B}(\vec{r})$ is a vector field, show, by explicit evaluation of the left- and right-hand sides in Cartesian coördinates, that

$$\vec{\nabla}\times(a\vec{B}) = (\vec{\nabla}a)\times\vec{B} + a(\vec{\nabla}\times\vec{B})\ . \tag{2.1}$$

b) Writing the "del operator" in spherical coördinates according to Symon's Eq. (3.124) allows us to write the curl of a vector as

$$\vec{\nabla}\times\vec{A} = \hat{r}\times\frac{\partial\vec{A}}{\partial r} + \frac{\hat{\theta}}{r}\times\frac{\partial\vec{A}}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\times\frac{\partial\vec{A}}{\partial\phi}\ . \tag{2.2}$$

Use this, along with Symon's Eq. (3.99), to calculate i) $\vec{\nabla} \times \hat{r}$; ii) $\vec{\nabla} \times \hat{\theta}$; iii) $\vec{\nabla} \times \hat{\phi}$.

c) Using the results of parts a) and b), and writing a vector field $\vec{A}(\vec{r})$ as

$$\vec{A}(\vec{r}) = A_r(r,\theta,\phi)\ \hat{r} + A_\theta(r,\theta,\phi)\ \hat{\theta} + A_\phi(r,\theta,\phi)\ \hat{\phi} \tag{2.3}$$

show that the curl in spherical coördinates is

$$\begin{aligned}\vec{\nabla} \times \vec{A} = &\left(\frac{1}{r}\partial_\theta A_\phi - \frac{1}{r\sin\theta}\partial_\phi A_\theta + \frac{\cos\theta}{r\sin\theta}A_\phi\right)\ \hat{r} + \left(\frac{1}{r\sin\theta}\partial_\phi A_r - \partial_r A_\phi - \frac{1}{r}A_\phi\right)\ \hat{\theta} \\ &+ \left(\partial_r A_\theta - \frac{1}{r}\partial_\theta A_r + \frac{1}{r}A_\theta\right)\ \hat{\phi}\end{aligned} \tag{2.4}$$

3 Force, Potential and Torque

Consider the force field

$$\vec{F}(\vec{r}) = V_0\, \frac{x\,\hat{x} + y\,\hat{y}}{x^2 + y^2} \tag{3.1}$$

a) By explicitly calculating the (three-dimensional) curl $\vec{\nabla} \times \vec{F}$, verify that this is a conservative force.

b) Obtain expressions for $\hat{x}$, $\hat{y}$ and $\hat{z}$ in terms of the cylindrical coördinates ρ, ϕ and z and the basis vectors $\hat{\rho}$, $\hat{\phi}$, and $\hat{z}$. (This can be done either by inverting Symon's Eq. (3.89) or directly from geometric considerations.) Simplify your answer as much as possible.

c) Use Symon's Eq. (3.87) and the results of part b) to write $\vec{F}$ above entirely in terms of the cylindrical coördinates ρ, ϕ and z and the basis vectors $\hat{\rho}$, $\hat{\phi}$, and $\hat{z}$ (and the constant V_0). Simplify your answer as much as possible.

d) Working in cylindrical coördinates, find the potential energy $V(\rho,\phi,z)$ such that $\vec{F} = -\vec{\nabla}V$. Include in your result an arbitrary constant (so that you capture the entire family of possible potentials) and indicate its units.

e) Calculate the vector torque $\vec{N}$ due to this force (in either Cartesian or cylindrical coördinates), and verify that the torque about the z axis vanishes.