Physics A300: Classical Mechanics I

Problem Set 6

Assigned 2004 October 26
Due 2004 November 2

Show your work on all problems!

1 Work Done Along a Path

Consider the path parametrized by

$$x(s) = s\,x_0 \tag{1.1a}$$
$$y(s) = s\,y_0 \tag{1.1b}$$
$$z(s) = s\,z_0 \tag{1.1c}$$

where s ranges from 0 to 1. The position vector associated with this path is

$$\vec{r}(s) = \hat{x}\,x(s) + \hat{y}\,y(s) + \hat{z}\,z(s) \tag{1.2}$$

a) What are the position vectors $\vec{r}(0)$ and $\vec{r}(1)$ of the endpoints of this path?

b) Describe the path succinctly in words.

c) Calculate the derivative $\frac{d\vec{r}}{ds}$ of the position vector with respect to the parameter.

d) Suppose that a particle moves along this path while being acted on by a force field $\vec{F}(\vec{r})$ with components

$$F_x(x, y, z) = ay \tag{1.3a}$$
$$F_y(x, y, z) = ax + by^3 + cyz \tag{1.3b}$$
$$F_z(x, y, z) = bz^3 + cy^2 z \tag{1.3c}$$

i) Write the dot product $\vec{F}(\vec{r}(s)) \cdot \frac{d\vec{r}}{ds}$, using the trajectory (1.1) to substitute for x, y, and z and write your answer only as a function of s (and the constants a, b, c, x_0, y_0, and z_0).

ii) Calculate the work done by the force (1.3) on the particle as it moves along the path $\vec{r}(s)$ from $s = 0$ to $s = 1$. (Note that there must be other forces involved in the problem to keep the particle on this path, so Newton's second law is not really useful here.)

2 Velocity-Dependent Force in Three Dimensions

Consider a particle moving under the influence of the velocity-dependent force

$$\vec{F} = \vec{b} \times \vec{v} \tag{2.1}$$

where $\vec{b} = b\hat{z}$ is a constant vector.

a) Writing the velocity as $\vec{v} = v_x\,\hat{x} + v_y\,\hat{y} + v_z\,\hat{z}$, work out the explicit form of $\vec{F}$ in terms of b, v_x, v_y, v_z, and the unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$.

b) By considering the three components of Newton's Second Law $\vec{F} = m\frac{d\vec{v}}{dt}$, find explicit expressions for $\dot{v}_x$, $\dot{v}_y$ and $\dot{v}_z$ in terms of m, b, and the components of $\vec{v}$.

c) The expressions you obtained in part b) are a set of coupled linear first-order differential equations for the three quantities $v_x(t)$, $v_y(t)$, and $v_z(t)$. The general solution will have a total of three arbitrary constants. You can solve them as follows:

   i) Take the time derivative of the equation for $\dot{v}_x$ to obtain an expression for $\ddot{v}_x$ in terms of $\dot{v}_y$.

   ii) Substitute the expression for $\dot{v}_y$ from part b) into the equation for $\ddot{v}_x$. This should give you a second order linear differential equation involving only v_x and its derivatives.

   iii) Find the general solution to the equation from part c)ii), which will give you an expression for v_x involving two arbitrary constants.

   iv) Take the time derivative of the result from part c)iii) to obtain an expression for $\dot{v}_x$, substitute this into the left-hand side of the first differential equation from part b), and solve algebraically for v_y.

   v) Solve the last differential equation from part b) to obtain an expression for v_z involving a third arbitrary constant.

d) Suppose the particle has initial velocity $\vec{v}(0) = v_0\,\hat{x}$. Use this initial condition to find the values of the three arbitrary constants in your general solution and write the form of $\vec{v}(t)$ in the presence of the force (2.1) given this initial condition.

e) If the particle is initially at position $\vec{r}(0) = x_0\,\hat{x} + y_0\,\hat{y} + z_0\,\hat{z}$, integrate the expressions for the components of $\vec{v}(t)$ to find the trajectory $\vec{r}(t) = x(t)\,\hat{x} + y(t)\,\hat{y} + z(t)\,\hat{z}$.

3 Conversion to Polar Coördinates

Converting a two-dimensional vector field from Cartesian to polar coördinates requires application not only of the coördinate transformations

$$x = r\cos\phi \tag{3.1a}$$
$$y = r\sin\phi \tag{3.1b}$$

but also the definitions of the basis vectors adapted to the two coördinate systems:

$$\hat{x} = \hat{r}\,\cos\phi - \hat{\phi}\,\sin\phi \tag{3.2a}$$
$$\hat{y} = \hat{r}\,\sin\phi + \hat{\phi}\,\cos\phi \tag{3.2b}$$

a) Show explicitly starting from (3.2) that

$$\hat{x}\,\cos\phi + \hat{y}\,\sin\phi = \hat{r} \tag{3.3a}$$
$$-\hat{x}\,\sin\phi + \hat{y}\,\cos\phi = \hat{\phi} \tag{3.3b}$$

b) Convert the following vector fields into polar coördinates. As an example, the vector field $\vec{F} = -kx\,\hat{x} - ky\,\hat{y}$ would be written $\vec{F} = -kr\,\hat{r}$.

   i) $\vec{F} = -kx\,\hat{x}$

   ii) $\vec{F} = -kx\,\hat{y} + ky\,\hat{x}$

   iii) $\vec{F} = -\frac{\alpha}{x^2+y^2}\,(x\,\hat{x} + y\,\hat{y})$