Physics A300: Classical Mechanics I

Problem Set 3

Assigned 2004 September 23
Due 2004 September 30

Show your work on all problems! Note that the answers to the some problems may be in the appendix of Symon, but they should only be used to check your work, since the answers alone are an insufficient solution to the problem.

Instructions to find y as a function of or in terms of a, b, and c should *not* be taken as an assertion that a, b, and c will all appear in the correct answer.

1 Motion Near a Stable Equilibrium

Consider a particle of mass m subject to the position-dependent force

$$F(x) = -\frac{mgx}{\sqrt{\ell^2 - x^2}} \tag{1.1}$$

where $-\ell < x < \ell$

a) Construct the potential energy $V(x)$ for this force, choosing the integration constant so that $V(0) = 0$.

b) By taking appropriate derivatives of $V(x)$, show that $x = 0$ is a *stable* equilibrium point.

c) Expand $V(x)$ as a power series in x, keeping terms out to x^2. You should end up with an approximation of the form

$$V(x) \approx V_q(x) = \frac{1}{2}kx^2 \tag{1.2}$$

only in place of k you will have some constant expression involving one or more of m, g, and ℓ.

d) Use your favorite computer plotting program to plot both $V(x)/mg\ell$ and $V_q(x)/mg\ell$ versus x/ℓ, on the same axes, for $x \in (-\ell, \ell)$. Turn in both the plot and the commands used to produce it.

e) Work out the (angular) frequency for small oscillations of the particle about the stable equilibrium $x = 0$ (written in terms of the constants m, g, and ℓ).

2 Double-Angle Formulas and the Euler Relation

Use the Euler relation $e^{i\theta} = \cos\theta + i\sin\theta$ to expand out

$$e^{i2\alpha} = \left(e^{i\alpha}\right)^2$$

in terms of sines and cosines. By requiring the real and imaginary parts of the resulting complex equation to hold separately, derive the double angle formulas for $\cos 2\alpha$ and $\sin 2\alpha$.

3 Energy in the Simple Harmonic Oscillator

Consider a particle of mass m moving in the potential $V(x) = \frac{1}{2}m\omega_0^2 x^2$, whose trajectory is

$$x(t) = A\cos(\omega_0 t + \phi) \tag{3.1}$$

a) Using the explicit solution (3.1), find

   i) The potential energy $V(t) = V(x(t))$ as a function of time in terms of A, ω_0, and ϕ.

   ii) The kinetic energy $T(t) = \frac{1}{2}m\dot{x}^2(t)$ as a function of time in terms of A, ω_0, and ϕ.

   iii) The total energy $E(t) = V(t) + T(t)$ as a function of time in terms of A, ω_0, and ϕ; show (by simplifying the resulting expression) that it is actually constant.

b) What is the period T with which the trajectory (3.1) repeats itself?

c) The notation $\langle\cdot\rangle$ indicates the time average of a quantity; since everything to do with the simple harmonic oscillator is periodic, it is reasonable to define this average over one period:

$$\langle f(t)\rangle = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,dt$$

where t_0 is some arbitrary starting time.

   i) Using the results of part a), calculate the three time averages $\langle V(t)\rangle$, $\langle T(t)\rangle$, and $\langle E(t)\rangle$. (You should find that your answers are independent of t_0, which is a consequence of the periodicity of the solution.)

   ii) Show *explicitly* from the results to part c)i) that $\langle V(t)\rangle + \langle T(t)\rangle = \langle E(t)\rangle$.