A Numerical Study of Relativistic Fluid Collapse

Final Defense

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Outline

- Theoretical Model of Non-equilibrium Neutron Stars
- Methods for their Numerical Simulation
- Parameter Space Survey and Dynamical Scenarios
  - Type I Critical Behavior
  - Type II Critical Behavior
- Conclusion
Theoretical Setting

\[ f = f(r, t) \]

- Dynamic, spherically-symmetric systems
Theoretical Setting

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\[ T_{ab} = (\rho_0 + \rho_0 \epsilon + P) u_a u_b + P g_{ab} \]

- Dynamic, spherically-symmetric systems
- Perfect fluid = isotropic fluid
  - Inviscid
  - No heat conduction
Theoretical Setting

\( f = f(r, t) \)

\( T_{ab} = (\rho + \rho \epsilon + P) u_a u_b + P g_{ab} \)

\( g_{ab} = -\alpha^2 dt^2 + a^2 dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \)

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- Polar-areal metric
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\[ g_{ab} = -\alpha^2 dt^2 + a^2 dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

\[ G_{ab} = 8\pi T_{ab} \]

- Dynamic, spherically-symmetric systems
- Perfect fluid = isotropic fluid
  - Inviscid
  - No heat conduction
- Polar-areal metric
- Time-dependent spacetime governed by Einstein’s Eq.
Fluid Equations of Motion

Local Conservation of Baryons Equation: \( \nabla_\mu J^\mu = 0 \)

Local Conservation of Energy Equation: \( \nabla_\mu T^{\mu \nu} = 0 \)

\[
\frac{\partial}{\partial t} \begin{bmatrix} D \\ S \\ \tau \\ q \end{bmatrix} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\alpha}{a} \begin{bmatrix} Dv \\ Sv + P \\ v (\tau + P) \end{bmatrix} \right) = \begin{bmatrix} 0 \\ \Sigma \\ 0 \\ \psi \end{bmatrix}
\]

\( v = \frac{au^r}{\alpha u^t}, \quad W^2 = \frac{1}{1 - v^2}, \quad D = a \rho_0 W, \quad S = (\rho + P) W^2 v, \quad \tau = S/v - D - P \)

- \( \Sigma = \Sigma(\alpha, a, q) \neq \Sigma(\alpha, a, q, \partial_r q, \partial_t q) \Rightarrow \text{EOM are hyperbolic!} \)
- Relativistic Ideal gas Equation of State: \( P = (\Gamma - 1) \rho_0 \epsilon, \quad \Gamma = \text{constant} \)
Slicing Condition:
\[
\frac{\alpha'}{\alpha} = a^2 \left[ 4\pi r (Sv + P) + \frac{1}{2r} \left( 1 - \frac{1}{a^2} \right) \right]
\]

Hamiltonian Constraint:
\[
\frac{a'}{a} = a^2 \left[ 4\pi r (\tau + D) - \frac{1}{2r} \left( 1 - \frac{1}{a^2} \right) \right]
\]

Mass Aspect Function:
\[
m(r, t) = \frac{r}{2} \left( 1 - \frac{1}{a^2} \right)
\]

Mass of Spherical Shell:
\[
\frac{dm}{dr} = 4\pi r^2 (\tau + D)
\]
Tolman-Oppenheimer-Volkoff (TOV) solutions:
Static, spherical solutions to Einstein’s Eq. w/ perfect fluid;

\[ \rho_0(r=0) = 0.05, \Gamma = 2 \]
Neutron Star Model

- Tolman-Oppenheimer-Volkoff (TOV) solutions:
  Static, spherical solutions to Einstein’s Eq. w/ perfect fluid;

- Parameterized by $\rho_c = \rho_0(0, 0)$
Tolman-Oppenheimer-Volkoff (TOV) solutions:
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Stable & Unstable Solutions
Neutron Star Model

Tolman-Oppenheimer-Volkoff (TOV) solutions:
Static, spherical solutions to Einstein’s Eq. w/ perfect fluid;

Parameterized by \( \rho_c = \rho_\odot(0, 0) \)

Stable & Unstable Solutions

Isentropic State Equations:
\[
P = K \rho_\odot^\Gamma, \quad P = (\Gamma - 1) \rho_\odot \epsilon
\]
\[
\Gamma = 2
\]
Velocity-Driven Collapse

Solve TOV Eq.'s
\[ \dot{g}_{\mu\nu} = \dot{T}_{\mu\nu} = 0 \]

Add in-going coordinate velocity:

\[ U \quad (r = R) = u_r u_t = p r r^2 b \]

Match to \( U = 0 \)

Solve \( a_0 \) and \( u_0 \)

and find \( v = a U \)

Tune to vary amount of kinetic energy
Velocity-Driven Collapse

Solve TOV Eq.’s \( \ddot{g}_{\mu\nu} = \dot{T}_{\mu\nu} = v = 0 \)

Add in-going coordinate velocity:
\[
U(\tilde{r} = r/R_*) = \frac{u^r}{u^t} = p \tilde{r} (\tilde{r}^2 - b)
\]
Velocity-Driven Collapse

Solve TOV Eq.'s ($\dot{g}_{\mu\nu} = \dot{T}_{\mu\nu} = v = 0$)

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\]

Match to \( U = 0 \)

Solve \( (\alpha' = \ldots) \) and \( (a' = \ldots) \)

and find \( v = aU/\alpha \)
Velocity-Driven Collapse

- Solve TOV Eq.'s \( \dot{g}_{\mu\nu} = \dot{T}_{\mu\nu} = v = 0 \)
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- Solve \( (\alpha' = \ldots) \) and \( (a' = \ldots) \)
  and find \( v = aU/\alpha \)
- Tune to vary amount of kinetic energy
Initial Data: TOV Solution

\[ a = (g_{rr})^{1/2} \]

\[ \alpha = (-g_{tt})^{1/2} \]

\[ \ln(\rho_\circ) \]

\[ \ln(P) \]
Initial Data: TOV + In-going Velocity

\[ a = (g_{rr})^{1/2} \]

\[ \alpha = (-g_{tt})^{1/2} \]

\[ \ln(\rho_\ast) \]

\[ \ln(P) \]
Einstein-massless-Klein-Gordon (EMKG) scalar field

\[ T_{ab}^{\text{scalar}} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} (\nabla_c \phi \nabla^c \phi) \]

\[ \nabla^a \nabla_a \phi = 0 \]

Coupled only through the geometry

\[ T_{ab} = T_{ab}^{\text{scalar}} + T_{ab}^{\text{fluid}} \quad , \quad G_{ab} = 8\pi T_{ab} \]

\[ \frac{dm}{dr} = \frac{dm_{\text{scalar}}}{dr} + \frac{dm_{\text{fluid}}}{dr} \]
High-resolution shock-capturing methods:
- Conservative, finite volume methods, e.g. solves differences of integral equations;
- Shocks propagate at correct speeds;
- Resolve shocks with very little Gibbs phenomenon near discontinuities;
- 2nd-order accuracy in smooth regions;

Adaptive, non-uniform discretization:
- $\Delta r(r) \propto e^r \rightarrow$ concentrates points near origin;
- Automatically adds points near origin when needed;
Primitive Variable Calculation:

\[ D = a \rho \omega W \]

\[ S = (\rho \omega + \rho \omega \epsilon + P) W^2 v \]

\[ \tau = S/v - D - P \]

Solve for \( P, v, \rho \omega \)

→ Finding minimum of non-linear function

\[
\text{Err}(w) = \ln \left[ \frac{w_{\text{calc}} - w_{\text{exact}}}{w_{\text{exact}}} \right]
\]

\[ \text{Err}(P), \text{Err}(v), \text{Err}(\rho \omega) \]
New formulation of fluid equations of motion:

\[ \Pi = \tau + S, \quad \Phi = \tau - S \]

Formulation improves accuracy of \( \tau \pm S \)
since \( \tau \to |S| \) as \( |v| \to 1 \)

Smoothing about sonic point in Type II collapse:

- Instability sets in as expansion shock develops;
- Dissipation subdues instability at discontinuity;
- Smoothing = Point-wise, nearest-neighbor averaging;
Parameter Space Survey

Previous work:
- S. Shapiro and Teukolsky (1980)
- Gourgoulhon (1992)
- Novak (2001)

Parameterized by $v_{\text{min}}$ and $\rho_c$

Dynamical scenarios:
- Normal Oscillations (O)
- Shock/Bounce/Oscillations (SBO)
- Shock/Bounce/Dispersal (SBD)
- Shock/Bounce/Collapse (SBC)
- Prompt Collapse (PC)
Normal Oscillations (O)

- All stars, small $v_{\text{min}}$
- Perturbed stable solution
- Normal, radial oscillations
Normal Oscillations (O)

- All stars, small $v_{\text{min}}$
- Perturbed stable solution
- Normal, radial oscillations
- Movies:
  \[ \ln (\rho_c(r, t)), \quad \rho_c(r, t), \quad v(r, t) \]
Shock/Bounce/Oscillations (SBO)

- Moderately compact stars, intermediate $v_{\text{min}}$
- Bounce, Core’s Rebound $\rightarrow$ Mass Ejection
- Highly-damped oscillations about sparser star
Moderately compact stars, intermediate $v_{\text{min}}$

Bounce, Core’s Rebound $\rightarrow$ Mass Ejection

Highly-damped oscillations about sparser star

Movies:

\[
\ln \left( \rho_0 \right) \& \ln \left( \epsilon \right) \, \text{vs.} \, \left\{ \ln \left( r/R_* \right), t \right\},
\]

\[
v \, \text{vs.} \, \left\{ \ln \left( r/R_* \right), t \right\}
\]
Sparse stars, small—to—large $v_{\text{min}}$

Bounce, Core’s Rebound $\rightarrow$ Dispersal

Negligible mass left behind
Sparse stars, small—to—large $v_{\text{min}}$

Bounce, Core’s Rebound $\rightarrow$ Dispersal

Negligible mass left behind

Movies:

$$\ln (\rho_0) \text{ vs. } \{\ln (r/R_*), t\},$$

$$v \text{ vs. } \{\ln (r/R_*), t\}$$
Sparse—to—semi-dense stars, medium—to—large $v_{\text{min}}$

- Bounce $\rightarrow$ Mass Ejection

- Black hole formation, $M_{\text{BH}} < M_{\ast}$
Sparse—to—semi-dense stars, medium—to—large $v_{\text{min}}$

Bounce $\rightarrow$ Mass Ejection

Black hole formation, $M_{\text{BH}} < M_{\ast}$

Movies:

$$a(r, t), \alpha(r, t), \rho(r, t), v(r, t)$$

$$r \in [0, R_{\ast}]$$
**Prompt Collapse (PC)**

- Nearly all stars, large $v_{\text{min}}$
- No mass ejection
- Black hole formation, $M_{\text{BH}} \sim M_*$
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- No mass ejection
- Black hole formation, $M_{\text{BH}} \sim M_*$
- Movies:

$$a(r, t), \alpha(r, t), \rho_\circ(r, t), v(r, t)$$

$$r \in [0, R_*]$$
Type I Critical Phenomena

\[ \rho_c = 0.15 \]

- Hawley & Choptuik (2000): Boson Stars
- Vary \( p \):
  \[ \phi(r, 0) = p \exp \left( - \frac{[r - r_\circ]^2}{\Delta^2} \right) \]
- Large \( p \) \( \rightarrow \) BLACK HOLE
- Small \( p \) \( \rightarrow \) NO BLACK HOLE
  (e.g. perturbed star)
- Tuning away the only unstable mode

\[ T_0 \propto \frac{1}{\omega_{Ly}} \ln |p - p^*| \]
$\rho_c = 0.15$

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- Small $p \rightarrow$ NO BLACK HOLE (e.g. perturbed star)
- Tuning away the only unstable mode
  \[ \Rightarrow T_0 \propto \frac{1}{\omega_{Ly}} \ln |p - p^*| \]
- Movies:
  \[ dm/dr, \ln(dm/dr) \text{ (wide view)}, \ln(dm/dr) \text{ (closeup)} \]
Type I: Anomalous Case $\rho_c = 0.197$

$\rho_c = 0.197$

$\rho_c(0, t)$

Movie: $dm/dr$
Type I: Anomalous Case $\rho_c = 0.27$

$\rho_c = 0.27$

$\rho_0(0, t)$

Movies: $dm/dr$, $\ln(dm/dr)$
Critical Solution = Unstable TOV ($\rho_c = 0.197$)
Expected scaling relationship:

\[ T_0 \propto \frac{1}{\omega_{Ly}} \ln |p - p^*| \]
Scaling Behavior

Expected scaling relationship:

\[ T_0 \propto \frac{1}{\omega_{Ly}} \ln |p - p^*| \]

\( \omega_{Ly} \propto \rho_c^* \)
Type II Critical Phenomena: Motivation

- “Ideal-gas” EOS: \( P = (\Gamma - 1) \rho \epsilon \), \( \Gamma = 2 \)
- Tuning star’s init. vel. \( \rightarrow \) Type II critical behavior;
- \( M_{BH} \propto |p - p^*|^\gamma \) with \( \gamma \simeq 0.52 \)

Neilsen and Choptuik (2000), Brady et al. (2002)
- Studied ultra-relativistic fluid collapse;
- A limit of “ideal-gas” case where \( \rho \equiv (1 + \epsilon) \rho_0 \simeq \rho_0 \epsilon \)
- \( P = (\Gamma - 1) \rho \), only EOS to admit CSS soln’s;
- For \( \Gamma = 2 \), \( \gamma \simeq 0.95 \pm 0.02 \)

Neilsen and Choptuik (2000)
- For \( \Gamma = 1.4 \): Ideal-gas Type II Sol’n. = Ultra-rel. Type II Sol’n.
Type II Critical Phenomena: Motivation


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Critical Regime of Parameter Space

- $T_{\text{max}} \equiv \text{Global Max.}(T^a_a)$

- $T^a_a = 3P - (\rho_0 + \rho_0 \epsilon)$

- Anticipated subcritical scaling behavior:
  $$T_{\text{max}} \propto |p - p^*|^{-2\gamma} \quad \gamma = 1/\omega_L$$

- Novak tuned to $\ln |p^* - p| \simeq -7$
Comparison of dimensionless quantities:

- $\omega \equiv 4\pi r^2 a^2 \rho$
- $a = \sqrt{g_{rr}}$
- $v = \frac{au_r}{u_t} = \text{Eulerian Velocity}$

(\(u^\mu = \text{Fluid's 4-velocity}\))

Star: $\rho_c = 0.05$

Ultra-relativistic fluid:
Initial profile = Gaussian

CSS Solutions of Ideal-gas and Ultra-rel.
Scaling of $T_{\text{max}}$: Dependence on Fluid’s Floor

- Floor = $2.5 \times 10^{-19}$
- Floor = $2.5 \times 10^{-17}$
- Floor = $2.5 \times 10^{-15}$

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$p^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9427</td>
<td>0.46875367383</td>
</tr>
<tr>
<td>0.9436</td>
<td>0.46875350285</td>
</tr>
<tr>
<td>0.9470</td>
<td>0.4687516089</td>
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</tbody>
</table>

- Floor used to prevent $v \geq 1$ , $P, \rho_0 < 0$
- No significant effect;
Scaling of $T_{\text{max}}$: Different “Families”

- Original
  - $U = U_2$
  - $\rho_*(r=0) = 0.0531$

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<tr>
<td>0.9423</td>
<td>0.42990315097</td>
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<tr>
<td>0.9187</td>
<td>0.4482047429836</td>
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</tbody>
</table>

Suggests scaling is fairly independent of:
- Functional form of perturbation;
- Initial star configuration;
Scaling of $T_{\text{max}}$: Different Flux Functions

- Suggests scaling is independent of flux formula;
- Able to tune further with “Smoothed” Roe solver;

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<tr>
<td>0.9399</td>
<td>0.46876822118</td>
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### Comparison of Scaling Parameters

<table>
<thead>
<tr>
<th>Study</th>
<th>Fluid Type</th>
<th>Scaling Parameter</th>
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<tbody>
<tr>
<td>Noble and Choptuik</td>
<td>Ideal gas</td>
<td>$\gamma = 0.94 \pm 0.01$</td>
</tr>
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<td>Ultra-relativistic fluid</td>
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<td>Ideal gas</td>
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Conclusion

Parameter Space Survey:
- Illuminated possible dynamical scenarios
- Provided a backdrop for critical phenomena studies

Type I Behavior:
- Critical solutions $\sim$ perturbed unstable TOV solutions
- Found anticipated scaling behavior $T_0 \propto \frac{1}{\omega_{LY}} \ln |p - p^*|$
- $\omega_{LY} \propto \rho_c^*$

Type II Behavior:
- Ideal gas critical solution $\sim$ ultra-relativistic critical solution
- $\gamma_{\text{ideal}} \sim \gamma_{\text{ultra-rel}}$
Future Work

- **Type I Phenomena:**
  - Compare results to $\omega_{Ly}$ of unstable TOV growing modes
  - Axially-symmetric collapse, effect of rotation
  - How $\omega_{Ly}(\rho^*_c)$ varies with $\Gamma$
  - Dependence on EOS

- **Type II Phenomena:**
  - Realistic equation of state
  - Axially-symmetric critical behavior
  - Develop general adaptive mesh refinement methods for relativistic fluids
Supporting Agencies

- NSERC = National Sciences and Engineering Research Council of Canada
- CIAR = Canadian Institute for Advance Research
- CFI = Canada Foundation for Innovation
- BCKDF = British Columbia Knowledge Development Fund

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