THE INTERNAL SPIN ANGULAR MOMENTUM OF AN ASYMPTOTICALLY FLAT SPACETIME

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- Generators of symmetry include new contribution from internal gauge group
  - In tetrad, generator of internal Spin(3,1) group gives rise to spin

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$$H_{spin-spin} = 2G\,\frac{3\,(\boldsymbol{S_1}\cdot\hat{\boldsymbol{r}})\,(\boldsymbol{S_2}\cdot\hat{\boldsymbol{r}}) - \boldsymbol{S_1}\cdot\boldsymbol{S_2}}{r^3}$$

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- Symplectic form on spatial hypersurface is conserved on covariant phase space

$$\boldsymbol{\Omega} = \frac{1}{k}\int_\Sigma \star\boldsymbol{\delta}\omega \wedge \boldsymbol{\delta}(e\, e) + \frac{\alpha}{2}\int_\Sigma \boldsymbol{\delta}(\bar{\psi} \star e\, e\, e) \wedge \boldsymbol{\delta}\psi + \boldsymbol{\delta}\bar{\psi} \wedge \boldsymbol{\delta}(\star e\, e\, e\, \psi)$$

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$$\phi : Spin(3,1) \rightarrow SO(3,1)$$

Internal and External Lorentz Groups are "locked" to preserve tetrad (Rigid gauge t-forms)

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Rotation: $\{\bar{K}, \lambda\}$ $\quad J_{tot} = D(\bar{K}) + G(\lambda) = -\frac{1}{k}\int_{\Sigma} \iota_{\bar{K}}(\star e\, e)\,\omega - \frac{1}{k}\int_{\Sigma} \star\lambda\, e\, e$

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Orbit — Spin

- Agreement between Noether charges, asymptotic integrals, and Komar integral is obtained if and only if spin term is included
  - Under assumption of global tetrad isometry:

$$-\frac{1}{k} \int_{\partial\Sigma} \star [K, e]\, \omega - \frac{1}{k} \int_{\partial\Sigma} \star \lambda\, e\, e = Q_{\{\bar{K},\lambda\}}$$

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$$Q_{\{\bar{K},\lambda\}} = \frac{\alpha}{2} \int_\Sigma \mathcal{L}_{\bar{K}} \bar{\psi} \star e\, e\, e\, \psi - \bar{\psi} \star e\, e\, e\, \mathcal{L}_{\bar{K}} \psi - \bar{\psi} \{\lambda, \star e\, e\, e\} \psi \, .$$

Orbit — Spin

- Agreement between Noether charges, asymptotic integrals, and Komar integral is obtained if and only if spin term is included
  - Under assumption of global tetrad isometry:

$$-\frac{1}{k} \int_{\partial\Sigma} \star [K, e]\, \omega - \frac{1}{k} \int_{\partial\Sigma} \star \lambda\, e\, e = Q_{\{\bar{K},\lambda\}}$$

Orbit

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- Noether charges follow from spin-enlarged Poincare algebra
  - Internal and external Lorentz groups must be phased locked to get conserved total angular momentum (including spin)

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Orbit Spin

  - Can be rearranged to reproduce Komar integral

$$-\frac{1}{2k}\int_{\partial\Sigma} *d\widetilde{K} = Q_{\{\bar{K},\lambda\}} - \frac{\alpha}{2}\int_{\Sigma} \iota_{\bar{K}}\left(m\,\bar{\psi} \star e\,e\,e\,e\,\psi\right)$$

Asymptotically Flat Phase Space

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- Spacetime reduces to fiducial flat tetrad at asymptotic infinity
  - Expand tetrad in power series $e = {}^0e + {}^1e/\rho + {}^2e/\rho^2 + ...$
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- Action is explicitly finite, symplectic form well-defined and conserved
- Phase space is not too restrictive
  - Contains all familiar asymptotically flat solutions

Asymptotic Expansion of Gauss functional

- Expand Gauss functional for spin(3,1) generator: $\chi = {}^0\chi + \frac{{}^1\chi}{\rho} + \frac{{}^2\chi}{\rho^2} + ...$

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$$\begin{aligned} {}^0G &= -\frac{1}{k}\int_{S^2} \star{}^0\chi\,{}^0e\,{}^0e \\ {}^1G &= -\frac{1}{k}\int_{S^2} \rho^{-1}(2\star{}^0\chi\,{}^0e\,{}^1e + \star{}^1\chi\,{}^0e\,{}^0e) \\ {}^2G &= -\frac{1}{k}\int_{S^2} \rho^{-2}(\star{}^0\chi\,{}^1e\,{}^1e + 2\star{}^1\chi\,{}^0e\,{}^1e + \star{}^2\chi\,{}^0e\,{}^0e + 2\star{}^0\chi\,{}^0e\,{}^2e) \end{aligned}$$

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- Gauss constraint contains two terms, define one as spin, one as charge

$$G(\lambda) = Q({}^2\lambda) + S({}^0\lambda)$$

$$\begin{aligned} Q({}^2\lambda) &\equiv -\frac{1}{k\rho^2}\int_{S^2} \star{}^2\lambda\,{}^0e\,{}^0e \\ S({}^0\lambda) &\equiv -\frac{1}{k\rho^2}\int_{S^2} 2\star{}^0\lambda\,{}^0e\,{}^2e \end{aligned}$$

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- Recall that Spin(3,1) and SO(3,1) subgroups are "locked"

$$\begin{aligned}\phi : \mathfrak{spin}(3,1) &\rightarrow \mathfrak{so}(3,1)\\ \lambda^{\{\hat{I}\hat{J}\}} &\mapsto \phi(\lambda^{\{\hat{I}\hat{J}\}}) = \bar{K}^{\{\hat{I}\hat{J}\}}\end{aligned}$$

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\{J^{\hat I\hat J}_{tot}\,,\,Q\} &= 0\\
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(Spin-enlarged) Poincare invariant charge

Symmetry Algebra

$$\begin{aligned}\{L^{\hat I\hat J}\,,\,L^{\hat K\hat L}\} &= 2\eta^{\hat I[\hat K}L^{\hat L]\hat J} - 2\eta^{\hat J[\hat K}L^{\hat L]\hat I}\\ \{L^{\hat I\hat J}\,,\,P^{\hat K}\} &= 2\eta^{\hat K[\hat J}P^{I]}\\ \{P^{\hat I}\,,\,P^{\hat J}\} &= 0\end{aligned}$$

$$Spin(3,1)\otimes(SO(3,1)\ltimes\mathbb{R}^{3,1})$$

$$\begin{aligned}\{S^{\hat I\hat J}\,,\,S^{\hat K\hat L}\} &= 2\eta^{\hat I[\hat K}S^{\hat L]\hat J} - 2\eta^{\hat J[\hat K}S^{\hat L]\hat I}\\ \{S^{\hat I\hat J}\,,\,P^{\hat K}\} &= 0\\ \{S^{\hat I\hat J}\,,\,L^{\hat K\hat L}\} &= 0\end{aligned}$$

$$Spin(3,1)\otimes(SO(3,1)\ltimes\mathbb{R}^{3,1})$$

$$\begin{aligned}\{Q({}^2\lambda)\,,\,L^{\hat I\hat J}\} &= -Q([\lambda^{\hat I\hat J},{}^2\lambda])\\ \{Q({}^2\lambda)\,,\,S^{\hat I\hat J}\} &= +Q([\lambda^{\hat I\hat J},{}^2\lambda])\\ \{Q({}^2\lambda)\,,\,P^{\hat I}\} &= 0\\ \{Q({}^2\lambda_1)\,,\,Q({}^2\lambda_2)\} &= 0\end{aligned}$$

$$\begin{aligned}\{J^{\hat I\hat J}_{tot}\,,\,Q\} &= 0\\ \{P^{\hat I}\,,\,Q\} &= 0\end{aligned}$$

(Spin-enlarged) Poincare invariant charge

- Casimir Invariants:

$$C_2 \equiv -M^2 = P_{\hat I}P^{\hat I} \qquad C_4/C_2 \equiv W_{\hat I}W^{\hat I}/P_{\hat I}P^{\hat I} = S(S+1) \qquad W_{\hat I} = \frac{1}{2}\epsilon_{\hat I\hat J\hat K\hat L}P^{\hat J}(L^{\hat K\hat L}+S^{\hat K\hat L})$$

References

arXiv:0905.4529
The Internal Spin Angular Momentum of an Asymptotically Flat Spacetime
Andrew Randono, David Sloan

arXiv:0906.1385
Do Spinors Frame-Drag?
Andrew Randono

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References