

THE INTERNAL SPIN ANGULAR MOMENTUM OF AN ASYMPTOTICALLY FLAT SPACETIME

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- Generators of symmetry include new contribution from internal gauge group
 - In tetrad, generator of internal $\text{Spin}(3,1)$ group gives rise to spin

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$$H_{spin-spin} = 2G \frac{3(\mathbf{S}_1 \cdot \hat{\mathbf{r}})(\mathbf{S}_2 \cdot \hat{\mathbf{r}}) - \mathbf{S}_1 \cdot \mathbf{S}_2}{r^3}$$

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- Symplectic form on spatial hypersurface is conserved on covariant phase space

$$\Omega = \frac{1}{k} \int_{\Sigma} \star \delta \omega \wedge \delta(e e) + \frac{\alpha}{2} \int_{\Sigma} \delta(\bar{\psi} \star e e e) \wedge \delta \psi + \delta \bar{\psi} \wedge \delta(\star e e e \psi)$$

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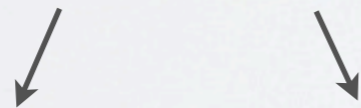
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Internal and External Lorentz Groups are “locked” to preserve tetrad (Rigid gauge t-forms)

$$\phi : Spin(3, 1) \rightarrow SO(3, 1)$$

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$$\text{Rotation: } \{\bar{K}, \lambda\} \quad J_{tot} = D(\bar{K}) + G(\lambda) = -\frac{1}{k} \int_{\Sigma} \iota_{\bar{K}}(\star e e) \omega - \frac{1}{k} \int_{\Sigma} \star \lambda e e$$

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- Agreement between Noether charges, asymptotic integrals, and Komar integral is obtained if and only if spin term is included
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- Noether charges follow from spin-enlarged Poincare algebra
 - Internal and external Lorentz groups must be phased locked to get conserved total angular momentum (including spin)

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Orbit Spin

- Can be rearranged to reproduce Komar integral

$$-\frac{1}{2k} \int_{\partial \Sigma} \star d\tilde{K} = Q_{\{\bar{K}, \lambda\}} - \frac{\alpha}{2} \int_{\Sigma} \iota_{\bar{K}} (m \bar{\psi} \star e e e e \psi)$$

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- Spacetime reduces to fiducial flat tetrad at asymptotic infinity
 - Expand tetrad in power series $e = {}^0e + {}^1e/\rho + {}^2e/\rho^2 + \dots$
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- Phase space is not too restrictive
 - Contains all familiar asymptotically flat solutions

Asymptotic Expansion of Gauss functional

- Expand Gauss functional for $\text{spin}(3,1)$ generator: $\chi = {}^0\chi + \frac{{}^1\chi}{\rho} + \frac{{}^2\chi}{\rho^2} + \dots$

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$${}^2G = -\frac{1}{k} \int_{S^2} \rho^{-2} (\star {}^0\chi {}^1e {}^1e + 2 \star {}^1\chi {}^0e {}^1e + \star {}^2\chi {}^0e {}^0e + 2 \star {}^0\chi {}^0e {}^2e)$$

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- Gauss constraint contains two terms, define one as spin, one as charge

$$G(\lambda) = Q({}^2\lambda) + S({}^0\lambda)$$

$$Q({}^2\lambda) \equiv -\frac{1}{k\rho^2} \int_{S^2} \star^2 \lambda^0 e^0 e$$

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(Spin-enlarged) Poincare
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$$\begin{aligned} \{S^{\hat{I}\hat{J}}, S^{\hat{K}\hat{L}}\} &= 2\eta^{\hat{I}[\hat{K} S^{\hat{L}]\hat{J}} - 2\eta^{\hat{J}[\hat{K} S^{\hat{L}]\hat{I}} \\ \{S^{\hat{I}\hat{J}}, P^{\hat{K}}\} &= 0 \\ \{S^{\hat{I}\hat{J}}, L^{\hat{K}\hat{L}}\} &= 0 \end{aligned}$$

$$Spin(3, 1) \otimes (SO(3, 1) \ltimes \mathbb{R}^{3,1})$$

$$\begin{aligned} \{Q(^2\lambda), L^{\hat{I}\hat{J}}\} &= -Q([\lambda^{\hat{I}\hat{J}}, ^2\lambda]) \\ \{Q(^2\lambda), S^{\hat{I}\hat{J}}\} &= +Q([\lambda^{\hat{I}\hat{J}}, ^2\lambda]) \\ \{Q(^2\lambda), P^{\hat{I}}\} &= 0 \\ \{Q(^2\lambda_1), Q(^2\lambda_2)\} &= 0 \end{aligned}$$

$$\begin{aligned} \{J_{tot}^{\hat{I}\hat{J}}, Q\} &= 0 \\ \{P^{\hat{I}}, Q\} &= 0 \end{aligned}$$

(Spin-enlarged) Poincare
invariant charge

- Casimir Invariants:

$$C_2 \equiv -M^2 = P_{\hat{I}} P^{\hat{I}} \quad C_4/C_2 \equiv W_{\hat{I}} W^{\hat{I}} / P_{\hat{I}} P^{\hat{I}} = S(S + 1) \quad W_{\hat{I}} = \frac{1}{2} \epsilon_{\hat{I}\hat{J}\hat{K}\hat{L}} P^{\hat{J}} (L^{\hat{K}\hat{L}} + S^{\hat{K}\hat{L}})$$

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The Internal Spin Angular Momentum of an Asymptotically Flat Spacetime

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[arXiv:0906.1385](#)

Do Spinors Frame-Drag?

[Andrew Randono](#)

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References