Spacetime and orbits of bumpy black holes

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Testing GR in the strong-field

- Observations of black holes are probing into the strong-field regime.
  - Orbits around Sgr A*
  - Spectrum of accretion disks
  - Pulsar timing (pulsar-BH binary)
  - Gravitational waves
- We need a phenomenological formulation of strong-field gravity tests.
  - For weak-field tests, we have the PPN formalism.
  - We want to construct a null experiment - define a set of parameters such that for GR, they are zero.

Why study bumpy black holes?

- Black holes provide a special opportunity to test GR thanks to the no-hair theorem.
  - The spacetime of a black hole is only a function of mass M and spin a.
  - The mass moments M_l and mass current moments S_l are related by $M_l + iS_l = M(ia)^l$.
- We need a framework for testing this, i.e. a black hole with "wrong" moments.
- Our proposal is to measure whether the deviation of black hole multipoles is zero.

Starting point: The Weyl solution

The Weyl solution is a general stationary, axisymmetric metric:

$$ds^2 = -e^{2\psi} dt^2 + e^{2\gamma - 2\psi} \left(d\rho^2 + dz^2\right) + e^{-2\psi} \rho^2 d\phi^2 \,.$$

We can put this in the form of the Schwarzschild metric for $\rho = r \sin\theta \sqrt{1 - \frac{2M}{r}}$, $z = (r - M)\cos\theta$, and

$$\psi = \frac{1}{2} \ln\left(1 - \frac{2M}{r}\right) \,,$$

$$\gamma = -\frac{1}{2} \ln\left(1 + \frac{M^2 \sin^2\theta}{r^2 - 2Mr}\right) \,.$$

Perturbing the Schwarzschild metric

Let $\psi = \psi_0 + \psi_1$, $\gamma = \gamma_0 + \gamma_1$. Define ψ_0 and γ_0 so that if $\psi_1 = 0$ and $\gamma_1 = 0$, we have the Schwarzschild metric:

$$\begin{aligned} ds^2 &= -e^{2\psi_1}\left(1-\frac{2M}{r}\right)dt^2 + e^{2\gamma_1-2\psi_1}\left(1-\frac{2M}{r}\right)^{-1}dr^2 \\ &\quad + e^{2\gamma_1-2\psi_1}r^2 d\theta^2 + e^{-2\psi_1}r^2\sin^2\theta d\phi^2 \,. \end{aligned}$$

To first order in ψ_1 and γ_1, the metric must satisfy the vacuum Einstein equations:

$$\frac{\partial^2\psi_1}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi_1}{\partial\rho} + \frac{\partial^2\psi_1}{\partial z^2} = 0 \quad \text{(Laplace's equation)}$$
$$\gamma_1 = f(\psi_1, \psi_0, \gamma_0)$$

Previous work: Collins and Hughes (2004)

Introduced the idea of perturbing a black hole instead of allowing for arbitrary higher moments

Looked at the effect of perturbations corresponding in the weak-field to a ring of mass and point masses at the poles

Limitations:

- It only looks at equatorial orbits and perturbations to Schwarzschild.
- The perturbation is not smooth, and calculating it requires performing numerical integrals.

Our approach

Consider smooth perturbations.

- Since ψ_1 must solve Laplace's equation in Weyl coordinates, we choose to look at perturbations for which ψ_1 is a spherical harmonic.

Consider a larger class of orbits and spacetimes:

- Consider inclined orbits.
- Create a way of adding a perturbation to Kerr.

The bumps cause changes in the geodesics, which we can measure by looking at precession frequencies.

Precession from the perturbation

We have three ways of calculating changes in precession frequencies:

- Numerical simulations
- Action-angle variables
- Canonical perturbation theory

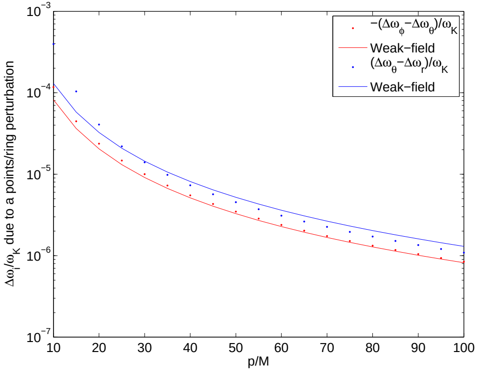
Action-angle variables and canonical perturbation theory

In action-angle formalism, periodic motion is described using angle variables w_i and action variables $J_i \equiv \oint p_i \, dq_i$.

The frequency of oscillation is given by $\omega_i \equiv 2\pi \frac{\partial H}{\partial J_i}$ where H is the Hamiltonian.

For a perturbed Hamiltonian $H + \Delta H$, to leading order the change in the frequency is $\Delta\omega_i = 2\pi \left. \frac{\partial \Delta H}{\partial J_i} \right|_0$.

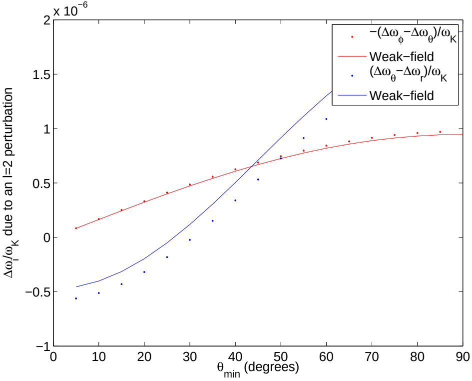
Orbits in a bumpy Schwarzschild spacetime



$$\Delta\omega_\phi - \Delta\omega_\theta \simeq -\omega_K \frac{9\mathcal{Q}\sin\theta_{\min}}{2p^2M^3}$$

$$\Delta\omega_\theta - \Delta\omega_r \simeq \omega_K \frac{9\mathcal{Q}(5\sin^2\theta_{\min}-1)}{4p^2M^3}$$

Orbits in a bumpy Schwarzschild spacetime



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Previous work: Glampedakis and Babak (2006)

Defines a ‘quasi-Kerr’ field

Consider deviation in the quadrupole moment: $\mathcal{Q} = \mathcal{Q}^{\mathrm{K}} - \epsilon M^3$

Quasi-Kerr metric: $g_{\mu\nu} = g^{\mathrm{K}}_{\mu\nu} + \epsilon h_{\mu\nu}$

Uses action-angle formalism to calculate frequencies

Quasi-Kerr field has Hamiltonian $H \equiv H_0 + \epsilon H_1$

New frequencies $\hat{\omega}_i \equiv \frac{\partial H}{\partial \hat{J}_i}$

Limitations:

- Only considers deviation from the quadrupole moment
- Equations of motion are only separable for equatorial orbits

From bumpy Schwarzschild to bumpy Kerr

Newman-Janis algorithm is a “rotation” in configuration space that transforms the Schwarzschild metric to the Kerr metric.

Applying the Newman-Janis algorithm to a bumpy Schwarzschild metric generates a bumpy Kerr metric.

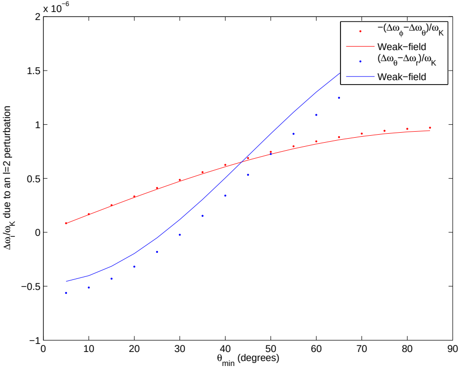
We can use canonical perturbation theory to calculate the precession for this spacetime.

Bumpy Kerr metric

$$
\begin{aligned}
ds^2 = & -e^{2\psi_1}\left(1-\frac{2Mr}{\Sigma}\right)dt^2-\left(e^{\gamma_1}-1\right)\frac{4a^2Mr\sin^2\theta}{\Delta\Sigma}dt\,dr \\
& -e^{-2\psi_1-\gamma_1}\frac{4aMr\sin^2\theta}{\Sigma}dt\,d\phi+e^{2\gamma_1-2\psi_1}\frac{\Sigma}{\Delta}dr^2 \\
& +\left(e^{\gamma_1}-1\right)\frac{2a\sin^2\theta}{\Sigma\Delta}\left[(r^2+a^2)^2-\Delta a^2\sin^2\theta\right]dr\,d\phi \\
& +e^{2\gamma_1-2\psi_1}\Sigma\,d\theta^2 \\
& +\left[e^{-2\psi_1}\frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)^2-\Delta a^2\sin^2\theta\right]\right. \\
& \qquad \left.+\left(e^{\gamma_1}-1\right)\frac{8a^2M^2r^2\sin^4\theta}{\Sigma(\Sigma-2Mr)}\right]d\phi^2
\end{aligned}
$$

where $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2Mr + a^2$

Orbits in a bumpy Kerr spacetime



$$\Delta\omega_\phi - \Delta\omega_\theta \simeq -\omega_K \frac{9\mathcal{Q}\sin\theta_{\rm min}}{2p^2 M^3}$$

$$\Delta\omega_\theta - \Delta\omega_r \simeq \omega_K \frac{9\mathcal{Q}(5\sin^2\theta_{\rm min} - 1)}{4p^2 M^3}$$

Conclusions

- We need to construct a null experiment to test gravity in the strong-field regime.
- Bumpy black holes let us test the no-hair theorem by changing the mass moments.
- These perturbations change the precession frequencies of geodesics.
- We can analytically calculate the effect of these perturbations using canonical perturbation theory.