The loop quantum cosmology of Bianchi I models

Edward Wilson-Ewing

Institute for Gravitation and the Cosmos
The Pennsylvania State University

Work with Abhay Ashtekar
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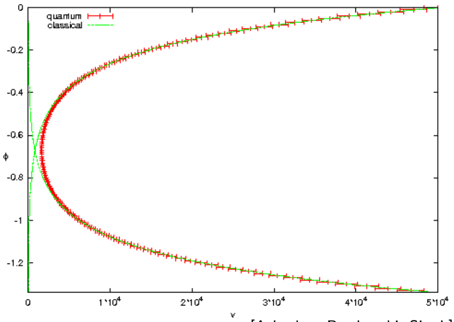
Outline

Loop quantum cosmology

Loop quantum cosmology takes the ideas of loop quantum gravity and applies them to simple, symmetry-reduced models like FRW cosmologies.

One of the more important results was to show that the big bang singularity is resolved due to quantum geometry effects in homogeneous and isotropic cosmologies. In addition, numerical simulations have shown that the singularity is replaced by a quantum bounce.



[Ashtekar, Pawlowski, Singh]

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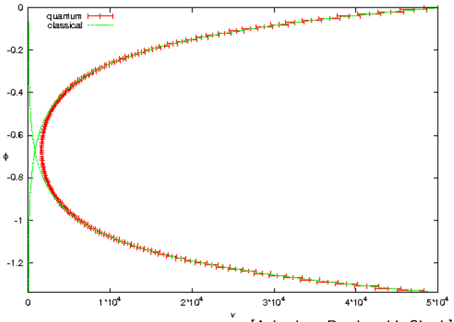

[Ashtekar, Pawlowski, Singh]

The obvious next step is to study models which allow anisotropies; Bianchi I cosmologies are the simplest such models.

The BKL conjecture

Quantum gravity effects are expected to be small at normal energy scales but should become large where general relativity fails, i.e. where singularities appear in the classical theory. For this reason, it is important to study the effects of quantum gravity near singularities.

In the 1970's, Belinskii, Khalatnikov and Lifshitz suggested that as a generic space-like singularity is approached, neighbouring points decouple from one another and each point can be described for long periods of time by the Bianchi I model. There has recently been a wealth of numerical results supporting their conjecture. [Andersson, Berger, Garfinkle, Moncrief, Rendall, ...]

Because of this, it is particularly interesting to study the loop quantum cosmology of Bianchi I models.

The Bianchi I space-time

The elementary variables in loop gravity are the densitized triad $E^a_i = \sqrt{q} e^a_i$ and the connection $A^i_a = \Gamma^i_a + \gamma K^i_a$.

Due to the symmetries of Bianchi I, they can each be parametrized by three variables:

$$E^a_i = p_i \left(\frac{\partial}{\partial x^i}\right)^a \qquad \text{and} \qquad A^i_a = c^i (dx^i)_a.$$

These are related to the variables in the metric

$$ds^2 = -N^2 dt^2 + a_1^2 dx_1^2 + a_2^2 dx_2^2 + a_3^2 dx_3^2,$$

as follows:

$$p_1 \propto a_2 a_3 \qquad \text{and} \qquad c_1 \propto \dot{a}_1 / N.$$

Classical Hamiltonian constraint

We will take the matter field to be a massless scalar field for two reasons: it is simple and the field ϕ can serve as a clock. Taking the symmetries of Bianchi I space-times into account, the Hamiltonian constraint is given by

$$\mathcal{C}_H = \int_{\mathcal{V}} \left[\frac{E^a_i E^b_j}{16\pi G \gamma^2} \epsilon^{ij}{}_k F_{ab}{}^k + \frac{p_\phi^2}{2} \right],$$

and, in terms of p_i and c_i, it is

$$\mathcal{C}_H = \int_{\mathcal{V}} \left[-\frac{1}{8\pi G \gamma^2} \left(p_1 p_2 c_1 c_2 + p_1 p_3 c_1 c_3 + p_2 p_3 c_2 c_3 \right) + \frac{p_\phi^2}{2} \right].$$

Finally, the only nonzero Poisson bracket is

$$\{c_i, p_j\} = 8\pi G \gamma \delta_{ij}.$$

Quantization procedure

In order to obtain the quantum theory, we must promote the variables to operators. This is easily done for p_i,

$$\hat{p}_1 |p_1, p_2, p_3\rangle = p_1 |p_1, p_2, p_3\rangle,$$

but it is more difficult for c_i as there is no operator corresponding to the connection in LQG. However, since there do exist operators corresponding to holonomies, it is possible to obtain an operator for ${F_{ab}}^k$ which is motivated by the classical relationship

$${F_{ab}}^k = -2 \lim_{Ar_\Box \to 0} \mathrm{Tr} \left(\frac{h_{\Box_{ij}} - 1}{Ar_\Box} \, \tau^k \right) (dx^i)_a \, (dx^j)_b .$$

However, it is impossible to take the limit of the area going to zero in the quantum theory as the area eigenvalues are discrete. Instead we will consider the case where $Ar_\Box$ is the smallest area eigenvalue $\Delta \ell_{\mathrm{Pl}}^2$ in LQC.

The $\hat{F}_{ab}{}^k$ operator

We find that

$$\hat{F}_{ab}{}^k = \sum_{i,j} \epsilon_{ij}{}^k \frac{\sin \bar{\mu}_i c_i \sin \bar{\mu}_j c_j}{\bar{\mu}_i \bar{\mu}_j} (dx^i)_a (dx^j)_b,$$

where $\bar{\mu}_1 = \sqrt{p_1 \Delta \ell_{\mathrm{Pl}}^2 / p_2 p_3}$. We then expand each $\sin \bar{\mu}_i c_i$ term into complex exponentials and we need to determine how $e^{i\bar{\mu}_i c_i}$ acts on a state.

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To do this, we introduce the variables $\lambda_i \propto \sqrt{p_i}$ and then we find

$$e^{i\bar{\mu}_i c_i} = \exp\left[-\frac{1}{\lambda_j \lambda_k} \frac{\partial}{\partial \lambda_i}\right].$$

This shows that the exponentials shift the wavefunctions:

$$e^{i\bar{\mu}_1 c_1} \Psi(\lambda_1, \lambda_2, \lambda_3) = \Psi(\lambda_1 - \tfrac{1}{\lambda_2 \lambda_3}, \lambda_2, \lambda_3).$$

We will also introduce $v \propto \lambda_1 \lambda_2 \lambda_3$.

Quantum Hamiltonian constraint

Combining all of the terms, we obtain a difference equation from the Hamiltonian constraint operator describing the quantum dynamics:

$$\partial_\phi^2 \Psi(\lambda_1, \lambda_2, v; \phi) = \frac{\pi G}{8} \sqrt{v} \Big[(v+2)\sqrt{v+4}\, F_4^+(\lambda_1, \lambda_2, v; \phi) - (v+2)\sqrt{v}\, F_0^+(\lambda_1, \lambda_2, v; \phi) - (v-2)\sqrt{v}\, F_0^-(\lambda_1, \lambda_2, v; \phi) + (v-2)\sqrt{v-4}\, F_4^-(\lambda_1, \lambda_2, v; \phi) \Big];$$

where

$$\begin{aligned} F_4^\pm(\lambda_1, \lambda_2, v; \phi) = {} & \Psi\left(\frac{v\pm 4}{v\pm 2}\cdot\lambda_1, \frac{v\pm 2}{v}\cdot\lambda_2, v\pm 4; \phi\right) + \Psi\left(\frac{v\pm 4}{v\pm 2}\cdot\lambda_1, \lambda_2, v\pm 4; \phi\right) \\ & + \Psi\left(\frac{v\pm 2}{v}\cdot\lambda_1, \frac{v\pm 4}{v\pm 2}\cdot\lambda_2, v\pm 4; \phi\right) + \Psi\left(\frac{v\pm 2}{v}\cdot\lambda_1, \lambda_2, v\pm 4; \phi\right) \\ & + \Psi\left(\lambda_1, \frac{v\pm 2}{v}\cdot\lambda_2, v\pm 4; \phi\right) + \Psi\left(\lambda_1, \frac{v\pm 4}{v\pm 2}\cdot\lambda_2, v\pm 4; \phi\right); \end{aligned}$$

$$\begin{aligned} F_0^\pm(\lambda_1, \lambda_2, v; \phi) = {} & \Psi\left(\frac{v\pm 2}{v}\cdot\lambda_1, \frac{v}{v\pm 2}\cdot\lambda_2, v; \phi\right) + \Psi\left(\frac{v\pm 2}{v}\cdot\lambda_1, \lambda_2, v; \phi\right) \\ & + \Psi\left(\frac{v}{v\pm 2}\cdot\lambda_1, \frac{v\pm 2}{u}\cdot\lambda_2, v; \phi\right) + \Psi\left(\frac{v}{v\pm 2}\cdot\lambda_1, \lambda_2, v; \phi\right) \\ & + \Psi\left(\lambda_1, \frac{v}{v\pm 2}\cdot\lambda_2, v; \phi\right) + \Psi\left(\lambda_1, \frac{v\pm 2}{v}\cdot\lambda_2, v; \phi\right). \end{aligned}$$

The singularity is resolved

A singular state is one which corresponds to a singular classical geometry. For Bianchi I cosmologies, this is the case iff $v = 0$. It can be shown that all singular states are annihilated by the gravitational part of the Hamiltonian constraint, that is

$$\partial_\phi^2 \Psi_{\mathrm{sing}} = 0.$$

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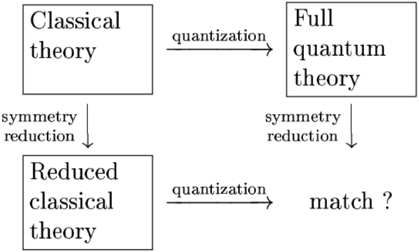
One can also show that a nonsingular state can never become a singular state under the action of the Hamiltonian constraint. In particular, if one starts with a wavefunction which is a superposition of only nonsingular states, the wavefunction will never have any support on singular states.

It is in this sense that the singularity is resolved.

Symmetry reduction

In LQC, symmetry reduced models are studied and progress can be made as these models are relatively simple. But can we symmetry reduce first and then quantize?



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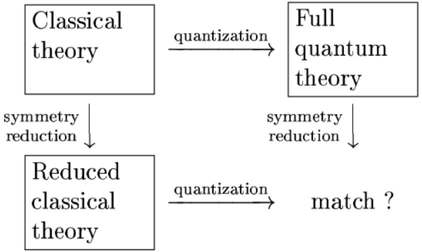

[Engle]

The projection $\Psi(v) = \sum_{\lambda_1,\lambda_2} \Psi(\lambda_1, \lambda_2, v)$ produces a wavefunction which depends only on the volume, not on any anisotropies. It can be shown that the dynamics for this wavefunction are identical to the dynamics for the flat isotropic FRW model in LQC.

$\Rightarrow$ This is an indication that it is ok to first symmetry reduce and to then quantize.

Conclusions

- Bianchi I space-times are particularly interesting to study in the context of quantum gravity due to the BKL conjecture
- The Hamiltonian constraint is well defined
- The singularity is resolved
- There are indications that quantizing symmetry reduced models is a viable approach
- Numerical studies are needed to fully understand the dynamics