Effective Constraints for Quantum Systems

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Introduction

- Classically constraints are conditions on phase-space $\Gamma_{\rm class}$
  - arise directly from the action principle
  - parts of $\Gamma_{\rm class}$ are unaccessible
  - some distinct points of $\Gamma_{\rm class}$ are physically equivalent
- Canonical quantization generally has to be modified for constraints –there is a number of methods with limited applicability
- "Effective" scheme for semiclassical states
  - enlarge $\Gamma_{\rm class}$ to $\Gamma_{\rm Q}$ adding leading order quantum parameters
  - formulate constraints for extra variables on $\Gamma_{\rm Q}$
  - analyze the enlarged system as classical

Why care about constrained systems?

- In short—general relativity is a constrained system
- Example: hamiltonian formulation (Arnowitt-Deser-Misner)
  - h_{ab} - 3-metric and its conjugate momentum π^{ab}
  - constraints have the form (appendix of Wald's book)

$$\frac{(16\pi G)^2}{\det h}\left(\pi^{ab}\pi_{ab} - \frac{1}{2}\pi^a{}_a\right) - {}^{(3)}\mathrm{R} = 0$$

$$D_a\left(\frac{\pi^{ab}}{\sqrt{\det h}}\right) = 0$$

- Symmetry reduced cosmological models are also constrained e.g. flat FRW universe scale factor a, conjugate momentum p_a

$$\frac{-2\pi G}{3}\frac{p_a^2}{a} + \mathscr{E}(a, p_a) = 0$$

Quantizing constraints

- Classically follow Dirac-Bergmann algorithm
  - solve constraints—restrict to region where they vanish
  - factor out gauge orbits
  - result: reduced phase-space Γ_{red}

- Γ_{red} generally not a cotangent bundle—no distinction between "configuration" and "momentum" variables
  $\longrightarrow$ ordinary quantization is undefined

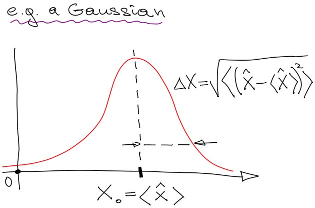
- Avoid this problem using Dirac's prescription
  - quantize the free system
  - promote constraints to operators $\hat{C}_{\mathrm{i}}$
  - impose $\hat{C}_{\mathrm{i}}|\psi_{\mathrm{phys}}\rangle = 0 \longleftarrow$ **difficult!**

- Exact implementations available only for special cases, cannot be perturbed directly

What "quantum parameters"?

- Quantum state of a particle in N-dimensions can be described by
  - $2N$ expectation values $\langle \hat{x}_\mathrm{i} \rangle$, $\langle \hat{p}_\mathrm{i} \rangle$
  - ∞ number of "moments"
    $\langle (\hat{x}_1 - \langle \hat{x}_1 \rangle)^{n_1} \dots (\hat{x}_\mathrm{N} - \langle \hat{x}_\mathrm{N} \rangle)^{n_N} (\hat{p}_1 - \langle \hat{p}_1 \rangle)^{m_1} \dots (\hat{p}_\mathrm{N} - \langle \hat{p}_\mathrm{N} \rangle)^{m_N} \rangle_{\mathrm{Weyl}}$

- For example $\langle (\hat{x}_\mathrm{i} - \langle \hat{x}_\mathrm{i} \rangle)^2 \rangle$ is the squared spread of the wave-function



- For semiclassical wave-functions "moments" $\propto \hbar^{\frac{1}{2}(\sum n_i + m_i)}$
  $\longrightarrow$ take lower order moments as "quantum parameters"
- Can be generalized to other quantum-mechanical systems

How do these parameters fit into Γ_{Q}?

- Γ_{class} comes with a Poisson bracket, crucial for dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}O = \{O, H\} + \frac{\partial}{\partial t}O$$

- Poisson structure on Γ_{Q} inspired by Ehrenfest's theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{O}\rangle = \frac{1}{i\hbar}\left\langle\left[\hat{O}, \hat{H}\right]\right\rangle + \frac{\partial}{\partial t}\langle\hat{O}\rangle$$

- Define $\left\{\langle\hat{A}\rangle, \langle\hat{B}\rangle\right\} := \frac{1}{i\hbar}\left\langle\left[\hat{A}, \hat{B}\right]\right\rangle$
brackets for moments follow from linearity and Leibniz rule

- $\langle\hat{H}\rangle$ generates quantum evolution, Schrödinger equation takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}X = \left\{X, \langle\hat{H}\rangle\right\} \rightarrow \ \infty \text{ number of coupled ODE} - \mathrm{s}$$

Implementing Dirac's prescription

- Physical states must satisfy $\hat{C}|\psi\rangle = 0$
- It follows $\langle\psi|\hat{C}|\psi\rangle = 0$
  $\rightarrow$ easy to enforce on quantum variables via $\langle\hat{C}\rangle = 0$
- Further, this implies $\langle\phi|\hat{C}|\psi\rangle = 0, \ \forall\ |\phi\rangle$. Involves two different states—not expressible in terms of moments directly
- For normalizable $|\psi\rangle$ and $|\phi\rangle$ there is some $\hat{A}$ s.t. $\langle\phi| = \langle\psi|\hat{A}$
- So we demand for all operators $\hat{A}$ polynomial in the basic observables

$$\boxed{\langle\hat{A}\hat{C}\rangle = 0}$$

Example: Newtonian Particle

- Free system two canonical pairs $\{\hat{x}, \hat{p}; \hat{t}, \hat{p}_t\}$, subject to $[\hat{x}, \hat{p}] = i\hbar = [\hat{t}, \hat{p}_t]$
- Observables constructed from polynomials in these basic elements
- Constraint has the form $\hat{C} = \hat{p}_t + \frac{\hat{p}^2}{2M} + V(\hat{x})$
- Systematically impose constraints order by order:

$$\left\langle \hat{x}^k \hat{p}^l \hat{t}^m \hat{p}_t^n \hat{C} \right\rangle = 0$$

- Infinitely many conditions—assume semiclassical state and truncate at some power of $\hbar^{\frac{1}{2}}$

Corrections of order $\hbar$

- Degrees of freedom: 4 expectation values $a = \langle \hat{a} \rangle$; 4 spreads $(\Delta a)^2 = \langle (\hat{a} - a)^2 \rangle$ and 6 covariances $\Delta(ab) = \langle (\hat{a} - a)(\hat{b} - b) \rangle_{\mathrm{Weyl}}$
- 5 non-trivial constraints left:

$$\langle \hat{C} \rangle = p_t + \frac{p^2}{2M} + \frac{(\Delta p)^2}{2M} = 0; \quad \langle \hat{p}\hat{C} \rangle = \Delta(pp_t) + \frac{p(\Delta p)^2}{M} = 0; \quad \langle \hat{p}_t \hat{C} \rangle = (\Delta p_t)^2 + \frac{p\Delta(pp_t)}{M} = 0;$$
$$\langle \hat{x}\hat{C} \rangle = \Delta(xp_t) + \frac{i\hbar p}{2M} + \frac{p\Delta(xp)}{M} = 0; \quad \langle \hat{t}\hat{C} \rangle = \frac{p\Delta(pt)}{M} + \Delta(tp_t) + \frac{i\hbar}{2} = 0.$$

- Four gauge freedoms remain, fix 3 of them:
  $\Delta(tp) = 0; \quad \Delta(xt) = 0; \quad (\Delta t)^2 = 0$
- In this gauge, evolution is generated by $\frac{p^2}{2M} + \frac{(\Delta p)^2}{2M} = \left\langle \frac{\hat{p}^2}{2M} \right\rangle$

Outlook

- Constructed a method for deriving semiclassical corrections for constrained quantum systems
- Applied to Newtonian and relativistic particle in a potential
- Cosmological models are to be analyzed next