Euclidean quantum gravity revisited

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Black hole thermodynamics

Black holes have thermal properties: consider e.g. the Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Variables

M mass and r radial distance from center

- Schwarzschild black hole has a temperature $T = 1/(8\pi M)$ and an entropy $\mathscr{S} = 4\pi M^2$; in general a black hole has a temperature $T = \kappa/(2\pi)$ and an entropy $\mathscr{S} = A/4$; κ is the surface gravity and A is the surface area
- Example: the entropy of a one solar-mass Schwarzschild black hole is $\mathscr{S} = 2.895 \times 10^{54} \quad \mathrm{J \cdot K^{-1}}$
- Quantum gravity will explain this entropy from first principles

Approaches to quantum gravity

As for any field theory, there are three different representations that can be employed to quantize gravity:

- Canonical approach $\longrightarrow$ loop quantum gravity
  - Background independent
  - Requires a space-time split
- Covariant approach $\longrightarrow$ perturbative string theory
  - Adapted to particle physics
  - Requires a fixed non-dynamical background
- Path integral approach $\longrightarrow$ Euclidean quantum gravity
  - Does not require a space-time split; does not require fixed background
  - Disadvantages? See below!

Overview: Path integrals in field theory

Recall from statistical mechanics the partition function

$$\mathcal{Z} = \mathrm{Tr}\left[\exp\left(-\beta\hat{H}[\phi]\right)\right] \quad \longrightarrow \quad \mathcal{Z} = \int \mathcal{D}[\phi]\exp\left(-\tilde{I}[\phi]\right)$$

Variables

ϕ fields, β inverse temperature, $\hat{H}$ Hamiltonian and $\tilde{I}$ Euclidean action

- Typically hard to evaluate $\mathcal{Z}$ exactly, so need approximation
- Standard trick for thermodynamics is to expand action around solutions ϕ_0 to the equations of motion $\delta\tilde{I} = 0$ and evaluate the *on-shell* partition function

$$\mathcal{Z} = \exp\left(-\tilde{I}[\phi_0]\right)$$

- Average energy $\langle E\rangle$ and entropy $\mathscr{S}$ can then be derived via:

$$\langle E\rangle = -\frac{\partial \ln \mathcal{Z}}{\partial\beta} \quad \text{and} \quad \mathscr{S} = \beta\langle E\rangle + \ln\mathcal{Z}$$

Metric-based actions for gravity: Gibbons-Hawking-York

The action for gravity on a manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$ in second-order form is given by:

$$\tilde{I}[g] = \frac{1}{2\kappa}\int_{\mathcal{M}} R d^D V + \frac{1}{\kappa}\oint_{\partial\mathcal{M}} (K - K_0) d^{D-1} V$$

Variables

$\kappa = 8\pi$ (with $G_D = 1$), R Ricci scalar of spacetime metric g and K trace of extrinsic curvature of boundary, $d^D V$ is volume element determined by g, and $d^{D-1} V$ is volume element determined by induced metric h on $\partial\mathcal{M}$

- For asymptotically flat spacetimes, the action is infinite, even for Minkowski spacetime itself
- Therefore ones adds the K_0 term to the boundary action, which is the extrinsic curvature of the boundary *embedded in flat spacetime*
- Resulting action is finite, but K_0 requires an isometric embedding into flat spacetime by definition and so the prescription cannot be applied to certain spacetimes

Metric-based actions for gravity: Mann-Marolf

A resolution to the problem is to define a new infinite counter-term that does not require an embedding at all.The resulting action, with Mann-Marolf counter-term $\hat{K}$ is given by

$$\tilde{I}[g] = \frac{1}{2\kappa}\int_{\mathcal{M}} R d^D V + \frac{1}{\kappa}\oint_{\partial\mathcal{M}} (K - \hat{K}) d^{D-1} V$$

- $\hat{K}$ is the trace of the tensor $\hat{K}_{ij}$, a local function of the boundary Ricci tensor $\hat{\mathcal{R}}_{ij}$, which is implicitly defined by solving the algebraic equation
$$\hat{\mathcal{R}}_{ij} = \hat{K}_{ij}\hat{K} - \hat{K}_i{}^k \hat{K}_{kj}$$
- This prescription is motivated by the Gauss-Codazzi equation
- Physically, it is desirable to employ a framework that generically produces finite quantities *without the need of adding any counter-terms*!

This leads us to consider...

First-order action

In the first-order formulation of general relativity the action is given by

$$\tilde{I}[e, A] = \frac{1}{4\kappa} \int_{\mathcal{M}} \epsilon_{IJKL} e^I \wedge e^J \wedge \Omega^{KL} - \frac{1}{4\kappa} \oint_{\partial\mathcal{M}} \epsilon_{IJKL} e^I \wedge e^J \wedge A^{KL}$$

Variables

e^I coframe, $A^I{}_J$ an $SO(4)$ connection, $\Omega^I{}_J$ associated curvature and ϵ_{IJKL} the totally antisymmetric Levi-Civita tensor

- Boundary term is the natural one on the configuration space $\mathcal{C} = \{e, A\}$ that is required by differentiablility
- Resulting action is both finite without the need of adding any counter-terms, and does not make any reference to the embedding of boundary in flat space
- Same boundary term in fact works for asymptotically anti-de Sitter spacetimes as well

Evaluation of the action: General considerations

- For black-hole spacetimes *in vacuum*, the bulk action is zero
- To evaluate the boundary terms, standard prescription is to evaluate seperately the contributions from the inner and outer boundaries by calculating the integrals on constant-r hypersurfaces and taking the limits as r goes to the horizon and to infinity; for all three examples considered below the contribution from the inner limit is zero
- In the first-order formalism, the calculation of τ_∞'s contribution amounts to calculating the ${}^2A$ contribution to the boundary integral, which can be obtained by expanding the co-frame in powers of r^{-1} and substituting the ${}^1e$ term in the equation

$$ {}^2A^{IJ} = 2r^2 \partial^{[J} \left(\frac{{}^1e^{I]}}{r} \right) $$

- The corresponding action becomes

$$ \tilde{I} = \frac{1}{\kappa} \oint_\infty {}^0{e_2}^2\, {}^0{e_3}^3\, \frac{{}^2{A_0}^{01}}{r^2} \partial_1 r $$

Example 1: Schwarzschild spacetime

Consider Euclidean Schwarzschild spacetime with line element as given by

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Note

Because time is now Euclidean the signature of the metric is $(++++)$ instead of $(-+++)$

- Regularity of the metric requires that τ have a period $\beta = 8\pi M$
- Action is evaluated to be $\tilde{I} = \frac{\beta^2}{16\pi}$
- Partition function is then $\mathcal{Z} = \exp[-\beta^2/(16\pi)]$
- Thermodynamic quantities are therefore $\langle E \rangle = M$ and $\mathscr{S} = 4\pi M^2 = A/4$

Example 2: NUT-charged spacetimes

Consider Euclidean Taub-NUT spacetime with line element given by

$$ds^2 = V(r)\,[d\tau + 2N\cos\theta d\phi]^2 + \frac{dr^2}{V(r)} + (r^2 - N^2)(d\theta^2 + \sin^2\theta d\phi^2)$$

Variables

$V(r) = (r^2 - 2Mr + N^2)/(r^2 - N^2)$ and N the NUT parameter

- $N = M$ is referred to as the "NUT" charge; $N = 4M/5$ is referred to as the "bolt" charge
- Regularity of the metric requires that τ have a period $\beta = 8\pi N$
- Action is evaluated to be $\tilde{I} = 4\pi MN$
- Partition function is then $\mathcal{Z} = \exp(-4\pi MN)$
- Substituting $M = N$ into $\mathcal{Z}$ we find $\langle E\rangle = N$ and $\mathscr{S} = 4\pi N^2$; substituting $M = 5N/4$ into $\mathcal{Z}$ we find $\langle E\rangle = 5N/4$ and $\mathscr{S} = 5\pi N^2$

Further work

Two directions are currently under investigation:

- Extend the formalism to asymptotically anti-de Sitter spacetimes; first-order boundary term is the same as for asymptotically flat spacetimes but asymptotics are different, therefore the partition function at ∞ will be different
- Look at stability of systems in first-order formalism: here we considered only the on-shell partition function. It would be of considerable interest to include the first quantum correction, i.e. quadratic term in the expansion of $\tilde{I}$. In the first-order framework there is an additional term in the action given by

$$\widetilde{H} = -\frac{1}{2\kappa\gamma}\int_{\mathcal{M}} e^I \wedge e^J \wedge \Omega_{IJ};$$

  γ is the Barbero-Immirzi parameter which does not show up in the equations of motion but does in the quadratic term. This may have important implications for the stability of asymptotically flat spacetimes

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