A Numerical Study of Boson Star Binaries

Bruno C. Mundim

with

Matthew W. Choptuik (UBC)

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General Motivation

• Why study compact binaries?
  • One of most promising sources of gravitational waves.
  • It is a good laboratory to study the phenomenology of strong gravitational fields.

• Why boson stars?
  • Matter similarities: Fluid stars and Boson stars have some similarity concerning the way they are modelled, e.g. both can be parametrized by their central density $\rho_0$ and have qualitatively similar plots of total mass vs $\rho_0$.
  • Then in the strong field regime for the compact binary system the dynamics may not depend sensitively on the details of the model.
  • Inspiral phases: Plunge and merge phase of the inspiral of compact objects is characterized by a strong dynamical gravitational field. In this regime gross features of fluid and boson stars’ dynamics may be similar.
  • Since the details of the dynamics of the stars (e.g. shocks) tend not to be important gravitationally, boson star binaries may provide some insight into NS binaries.
Matter Model: Scalar Field

- **Star-like solutions:** A massive complex field is chosen as matter source because it is a simple type of matter that allows a star-like solution and because there will be no problems with shocks, low density regions, ultrarelativistic flows, etc in the evolution of this kind of matter as opposed to fluids.

- **Static spacetimes:** *Complex* scalar fields allow the construction of static spacetimes. The matter content is then described by $\Phi = \phi_1 + i\phi_2$, where $\phi_1$ and $\phi_2$ are real-valued.

- **Equations of motion:** Klein-Gordon equation:

  $$\Box \phi_A - m^2 \phi_A = 0, \quad A = 1, 2.$$  

- **Hamiltonian Formulation:** In terms of the conjugate momentum field $\Pi_A$:

  $$\partial_t \phi_A = \frac{\alpha^2}{\sqrt{-g}} \Pi_A + \beta^i \partial_i \phi_A,$$  

  $$\partial_i \Pi_A = \partial_i (\beta^i \Pi_A) + \partial_i (\sqrt{-g} \gamma^{ij} \partial_j \phi_A) - \sqrt{-g} m^2 \phi_A.$$
Conformally Flat Approximation (CFA)

- Motivation

- Facts and assumptions:
  - Full 3D Einstein equations are very complex and computationally expensive to solve.
  - Gravitational radiation is small in most systems studied so far.
  - Heuristic assumption that the dynamical degrees of freedom of the gravitational fields, i.e. the gravitational radiation, play a small role in at least some phases of the strong field interaction of a merging binary.

- An approximation candidate:
  - CFA effectively eliminates the two dynamical degrees of freedom, simplifies the equations and allows a fully constrained evolution.
  - CFA allows us to investigate the same kind of phenomena observed in the full relativistic case, such as the description of compact objects and the dynamics of their interaction; black hole formation; critical phenomena.
Conformally Flat Approximation (CFA)

• **Formalism**
  
  • Einstein field equations cast into 3+1/ADM form.
  • The CFA prescribes a conformally flat spatial metric at all times.
  • Introduce a flat metric $f_{ij}$ as a base / background metric:
    \[
    \gamma_{ij} = \psi^4 f_{ij}, \tag{4}
    \]
    where the conformal factor $\psi$ is a positive scalar function describing the ratio between the scale of distance in the curved space and flat space ($f_{ij} \equiv \delta_{ij}$ in cartesian coordinates).
  
  • Maximum slicing condition is used to fix the time coordinate:
    \[
    K^i_i = 0, \quad \partial_t K^i_i = 0. \tag{5}
    \]
    
  • In this approximation all of the geometric variables can be computed from the constraints as well as from a specific choice of coordinates.
Conformally Flat Approximation (CFA)

- **Slicing Condition**
  - Gives an elliptic equation for the lapse function $\alpha$:

\[
\nabla^2 \alpha = -\frac{2}{\psi} \nabla \psi \cdot \nabla \alpha + \alpha \psi^4 \left( K_{ij} \hat{K}^{ij} + 4\pi (\rho + S) \right).
\]  
(6)

- **Hamiltonian Constraint**
  - Gives an elliptic equation for the conformal factor $\psi$:

\[
\nabla^2 \psi = -\frac{\psi^5}{8} \left( K_{ij} \hat{K}^{ij} + 16\pi \rho \right).
\]  
(7)

- **Momentum Constraints**
  - Given elliptic equations for the shift vector components $\beta^i$:

\[
\nabla^2 \beta^j = -\frac{1}{3} \hat{\gamma}^{ij} \partial_i \left( \nabla \cdot \vec{\beta} \right) + \alpha \psi^4 16\pi J^j - \partial_i \left[ \ln \left( \frac{\psi^6}{\alpha} \right) \right] \left[ \hat{\gamma}^{ik} \partial_k \beta^j \right.
\]
\[
\left. + \hat{\gamma}^{jk} \partial_k \beta^i - \frac{2}{3} \hat{\gamma}^{ij} \left( \nabla \cdot \vec{\beta} \right) \right].
\]  
(8)
Note that $K_{ij}K^{ij}$ can also be expressed in terms of the flat operators. It ends up being expressed as flat derivatives of the shift vector:

$$K_{ij}K^{ij} = \frac{1}{2\alpha^2} \left( \hat{\gamma}_{kn} \hat{\gamma}^{ml} \hat{D}_m \beta^k \hat{D}_l \beta^n + \hat{D}_m \beta^l \hat{D}_l \beta^m - \frac{2}{3} \hat{D}_l \beta^l \hat{D}_k \beta^k \right).$$ (9)
Then the following set of functions completely characterize the geometry at each time slice:

\[ \alpha = \alpha(t, \vec{r}), \quad \psi = \psi(t, \vec{r}), \quad \beta^i = \beta^i(t, \vec{r}), \quad (10) \]

where \( \vec{r} \) depends on the coordinate choice for the spatial hypersurface.

The solution of the gravitational system under CFA and maximal slicing condition can be summarized as:

- Specify initial conditions for the complex scalar field.
- Solve the elliptic equations for the geometric quantities on the initial slice.
- Update the matter field values to the next slice using their equation of motion.
- For the new configuration of matter fields, re-solve the elliptic equations for the geometric variables and again allow the matter fields to react and evolve to the next slice and so on.
Conformally Flat Approximation (CFA)

- Discretization Scheme:

\[ Lu - f = 0 \quad \Rightarrow \quad L^h u^h - f^h = 0. \]  \hspace{1cm} (11)

- For hyperbolic operators \( L \): second order accurate Crank-Nicholson scheme.
- For elliptic operators \( L \): second order accurate centred finite difference operators.
- Dirichlet Boundary conditions applied.

- Numerical Techniques:
  - pointwise Newton-Gauss-Seidel (NGS) iterative technique was used to solve the finite difference equations originated from the hyperbolic set of equations.
  - Full Approximation Storage (FAS) multigrid algorithm was applied on the discrete version of the elliptic set of equations. NGS is used in this context as a smoother of the solution error.
- Static spherically symmetric ansatz: $\phi(t, r) = \phi_0(r)e^{-i\omega t}$. 
Initial Data

- Family of static spherically symmetric solutions:

\[ M_{\text{ADM}} \times \phi_0(0) \]

\[ M_{\text{ADM}} \times R \]
Evolution Results

• Remarks: \( \rho \sim |\phi(t, r)|^2 = \phi_0(r)^2 \), and Planck units are adopted, i.e. \( G = c = \hbar = 1 \).

• Results summary:

  • **Initial data:**
    
    • Each boson star is modelled as a static, spherically symmetric solution of the Einstein-Klein-Gordon system.
    
    • They each have a central scalar field value of \( \phi_0(0) = 0.02 \), that corresponds to a boson star with radius of \( R_{99} \approx 17 \) and ADM mass of \( M_{ADM} \approx 0.475 \).
    
    • Each solution is then superposed and boosted in opposite directions (along \( x \) axis).

  • **Evolution:**
    
    • **Orbital motion and interrupted orbits:** The stars lie along the \( y \) axis with a coordinate separation between their centers of 40. Three distinct cases studied corresponding to different initial velocities:
      
      • \( v_x = 0.09 \): two orbital periods.
      
      • \( v_x = 0.07 \): rotating boson star as final merger.
      
      • \( v_x = 0.05 \): possible black-hole formation.

    • **Head-on collision:** Stars along \( x \) axis with coordinate separation of 50:
      
      • \( v_x = 0.4 \): solitonic behaviour.
• **Orbital Dynamics:** 2 boson stars - \( v_x = 0.09 \).

\[
Z = 0 \text{ slice for } |\phi|
\]

\[
Z = 0 \text{ slice for } \alpha
\]

• \( \phi_0 : 0.02 \). Physical coordinate domain: 120 per edge. Physical time: \( t = 4500 \). Simulation parameters: Courant factor \( \lambda = 0.4 \); Grid size: \( 113^3 \); 2.4GHz Dual-Core AMD Opteron CPU time: 285 hours (12 days).
**Evolution Results**

- **Orbital Dynamics:** 2 boson stars - $v_x = 0.07$.

\[ Z = 0 \text{ slice for } |\phi| \]

\[ Z = 0 \text{ slice for } \alpha \]

- $\phi_0 : 0.02$. Physical coordinate domain: 120 per edge. Physical time: $t = 2250$. Simulation parameters: Courant factor $\lambda = 0.4$; Grid size: $113^3$; 2.4GHz Dual-Core AMD Opteron CPU time: 158 hours (6.5 days).
Evolution Results

- **Orbital Dynamics:** 2 boson stars - $v_x = 0.05$.

$Z = 0$ slice for $|\phi|$

- $\phi_0 : 0.02$. Physical coordinate domain: 120 per edge. Physical time: $t = 1500$. Simulation parameters: Courant factor $\lambda = 0.4$; Grid size: $113^3$; 2.4GHz Dual-Core AMD Opteron CPU time: 115 hours (4.5 days).
Evolution Results

- Head-on collision: 2 boson stars - $v_x = 0.4$.

\[ Z = 0 \text{ slice for } |\phi| \]

\[ Z = 0 \text{ slice for } \alpha \]

- $\phi_0 : 0.02$. Physical coordinate domain: $[-50, 50, -25, 25, -25, 25]$. Total physical time: $t = 140$. Simulation parameters: $\lambda = 0.4$; Grid size: $[N_x, N_y, N_z] = [129, 65, 65]$;
Conclusion and Future Directions

• We were able to probe a few outcomes of the orbital dynamics of boson stars within the CFA.

• At least qualitatively, we observed a few characteristic phenomena of the fully relativistic case, such as orbital precession and scalar matter solitonic behaviour.

• These results are quite promising and suggest that, with enhancements such as the incorporation of AMR techniques and parallel execution capabilities, this code will be a powerful tool for investigating the strong gravity effects in the interaction of boson stars.

• We hope, in the future, to be able to calibrate the CFA fidelity to the fully general relativistic case and use this code to survey the parameter space of the orbital dynamics.
Appendix A - CFA equations of motion

- 3d Cartesian Coordinates

\[ \partial_t \phi_A = \frac{\alpha}{\psi^6} \Pi_A + \beta^i \partial_i \phi_A \]  
(12)

\[ \partial_t \Pi_A = \partial_x (\beta^x \Pi_A + \alpha \psi^2 \partial_x \phi_A) + \partial_y (\beta^y \Pi_A + \alpha \psi^2 \partial_y \phi_A) + \partial_z (\beta^z \Pi_A + \alpha \psi^2 \partial_z \phi_A) - \alpha \psi^6 \frac{dU(\phi_0^2)}{d\phi_0^2} \phi_A \]  
(13)

\[ \frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial^2 \alpha}{\partial y^2} + \frac{\partial^2 \alpha}{\partial z^2} = -\frac{2}{\psi} \left[ \frac{\partial \psi}{\partial x} \frac{\partial \alpha}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial \alpha}{\partial y} + \frac{\partial \psi}{\partial z} \frac{\partial \alpha}{\partial z} \right] + \alpha \psi^4 \left( K_{ij} K^{ij} + 4\pi (\rho + S) \right) \]  
(14)

\[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -\frac{\psi^5}{8} \left( K_{ij} K^{ij} + 16\pi \rho \right) \]  
(15)
Appendix A - CFA equations of motion

- $\times$ component of the shift vector in cartesian coordinates

$$\frac{\partial^2 \beta^x}{\partial x^2} + \frac{\partial^2 \beta^x}{\partial y^2} + \frac{\partial^2 \beta^x}{\partial z^2} = -\frac{1}{3} \frac{\partial}{\partial x} \left( \frac{\partial \beta^x}{\partial x} + \frac{\partial \beta^y}{\partial y} + \frac{\partial \beta^z}{\partial z} \right) + \alpha \psi^4 16\pi J^x$$

$$\begin{align*}
- \frac{\partial}{\partial x} \left[ \ln \left( \frac{\psi^6}{\alpha} \right) \right] & \left[ \frac{4 \partial \beta^x}{3 \partial x} - \frac{2}{3} \left( \frac{\partial \beta^y}{\partial y} + \frac{\partial \beta^z}{\partial z} \right) \right] \\
- \frac{\partial}{\partial y} \left[ \ln \left( \frac{\psi^6}{\alpha} \right) \right] & \left[ \frac{\partial \beta^x}{\partial y} + \frac{\partial \beta^y}{\partial y} \right] \\
- \frac{\partial}{\partial z} \left[ \ln \left( \frac{\psi^6}{\alpha} \right) \right] & \left[ \frac{\partial \beta^x}{\partial z} + \frac{\partial \beta^z}{\partial z} \right]
\end{align*}$$

(16)

- $K_{ij} K^{ij}$ in 3d cartesian coordinates

$$K_{ij} K^{ij} = \frac{1}{2\alpha^2} \left[ \left( \frac{\partial \beta^x}{\partial x} \right)^2 + \left( \frac{\partial \beta^x}{\partial y} \right)^2 + \left( \frac{\partial \beta^x}{\partial z} \right)^2 + \left( \frac{\partial \beta^y}{\partial x} \right)^2 + \left( \frac{\partial \beta^y}{\partial y} \right)^2 + \left( \frac{\partial \beta^y}{\partial z} \right)^2 \\
+ \left( \frac{\partial \beta^z}{\partial x} \right)^2 + \left( \frac{\partial \beta^z}{\partial y} \right)^2 + \left( \frac{\partial \beta^z}{\partial z} \right)^2 + \frac{\partial}{\partial x} \left( \beta^x \frac{\partial}{\partial x} + \beta^y \frac{\partial}{\partial y} + \beta^z \frac{\partial}{\partial z} \right) \beta^x \\
+ \frac{\partial}{\partial y} \left( \beta^x \frac{\partial}{\partial x} + \beta^y \frac{\partial}{\partial y} + \beta^z \frac{\partial}{\partial z} \right) \beta^y + \frac{\partial}{\partial z} \left( \beta^x \frac{\partial}{\partial x} + \beta^y \frac{\partial}{\partial y} + \beta^z \frac{\partial}{\partial z} \right) \beta^z \\
- \frac{2}{3} \left( \frac{\partial \beta^x}{\partial x} + \frac{\partial \beta^x}{\partial y} + \frac{\partial \beta^x}{\partial z} \right)^2 \right]$$

(17)
Appendix B: Boson Stars in Spherical Symmetry

- Spherically Symmetric Spacetime (SS):

\[ ds^2 = (-\alpha^2 + a^2 \beta^2) dt^2 + 2a^2 \beta dtdr + a^2 dr^2 + r^2 b^2 d\Omega^2 , \]  

\hspace{1cm} (18)

- Hamiltonian constraint:

\[ -\frac{2}{arb} \left\{ \left[ \frac{(rb)'}{a} \right]' + \frac{1}{rb} \left[ \left( \frac{rb}{a} (rb)' \right)' - a \right] \right\} + 4K^r_r K^\theta_\theta + 2K^\theta_\theta^2 = 8\pi \left[ \frac{|\Phi|^2 + |\Pi|^2}{a^2} + m^2 |\phi|^2 \right] \]  

\hspace{1cm} (19)

- Momentum constraint:

\[ K^\theta_\theta' + \frac{(rb)'}{rb} (K^\theta_\theta - K^r_r) = \frac{2\pi}{a} \left( \Pi^* \Phi + \Pi \Phi^* \right) . \]  

\hspace{1cm} (20)

where the auxiliary field variables were defined as:

\[ \Phi \equiv \phi' , \]  

\hspace{1cm} (21)

\[ \Pi \equiv \frac{a}{\alpha} \left( \dot{\phi} - \beta \phi' \right) , \]  

\hspace{1cm} (22)
Boson Stars in Spherical Symmetry

- **Evolution equations**

\[
\dot{a} = -\alpha a K^r_r + (a\beta)'
\]

\[
\dot{b} = -\alpha b K^\theta_\theta + \frac{\beta}{r} (rb)'.
\]

\[
\dot{K}^r_r = \beta K^{r'}_r - \frac{1}{a} \left( \frac{\alpha'}{a} \right)' + \alpha \left\{ -\frac{2}{arb} \left[ \frac{(rb)'}{a} \right]' + KK^r_r - 4\pi \left[ \frac{2|\Phi|^2}{a^2} + m^2|\phi|^2 \right] \right\}
\]

\[
\dot{K}^\theta_\theta = \beta K^{\theta'}_\theta + \frac{\alpha}{(rb)^2} - \frac{1}{a(rb)^2} \left[ \frac{\alpha rb}{a} (rb)' \right]' + \alpha \left( KK^\theta_\theta - 4\pi m^2|\phi|^2 \right)
\]

- **Field evolution equations**

\[
\dot{\phi} = \frac{\alpha}{a} \Pi + \beta \Phi
\]

\[
\dot{\Phi} = \left( \beta \Phi + \frac{\alpha}{a} \Pi \right)'
\]

\[
\dot{\Pi} = \frac{1}{(rb)^2} \left[ (rb)^2 \left( \beta \Pi + \frac{\alpha}{a} \Phi \right) \right]' - \alpha am^2 \phi + 2 \left[ \alpha K^\theta_\theta - \beta \frac{(rb)'}{rb} \right] \Pi
\]
Appendix B: Boson Stars in Spherical Symmetry

- Maximal-isotropic coordinates

- Maximal slicing condition

\[ K \equiv K^i_i = 0 \quad \dot{K}(t, r) = 0 \]  

- Isotropic condition

\[ a = b \equiv \psi(t, r)^2 \]  

- They fix the lapse and shift (equivalent of fixing the coordinate system)

\[ \alpha'' + \frac{2}{r \psi^2} \frac{d}{dr} \left( r^2 \psi^2 \right) \alpha' + \left[ 4 \pi \psi^4 m^2 \phi^2 - 8 \pi |\Pi|^2 - \frac{3}{2} \left( \psi^2 K^r_r \right)^2 \right] \alpha = 0 \]  

\[ r \left( \frac{\beta}{r} \right)' = \frac{3}{2} \alpha K^r_r \]  

- Constraint equations

\[ \frac{3}{\psi^5} \frac{d}{dr^3} \left( r^2 \frac{d \psi}{dr} \right) + \frac{3}{16} K^r_r \psi^2 = -\pi \left( \frac{|\Phi|^2 + |\Pi|^2}{\psi^4} + m^2 |\phi|^2 \right) \]  

\[ K^{r}{}_{r}{}' + 3 \frac{(r \psi^2)'}{r \psi^2} K^r_r = -\frac{4 \pi}{\psi^2} (\Pi^* \Phi + \Pi \Phi^*) \]
Appendix B: Boson Stars in Spherical Symmetry

- Complex-scalar field evolution equations

\[ \dot{\phi} = \frac{\alpha}{\psi^2} \Pi + \beta \Phi \] (36)

\[ \dot{\Phi} = \left( \beta \Phi + \frac{\alpha}{\psi^2} \Pi \right)' \] (37)

\[ \ddot{\Pi} = \frac{3}{\psi^4 \frac{d}{dr^3}} \left[ r^2 \psi^4 \left( \beta \Pi + \frac{\alpha}{\psi^2} \Phi \right) \right] - \alpha \psi^2 m^2 \phi \]

\[ - \left[ \alpha K^r r + 2 \beta \frac{(r \psi^2)'}{r \psi^2} \right] \Pi \] (38)
Appendix B: Boson Stars in Spherical Symmetry

- These equations were coded using RNPL and tested for a gaussian pulse as initial data.
Appendix B: Boson Stars in Spherical Symmetry

- **Initial Value Problem**

- We are interested in generating *static* solutions of the Einstein-Klein-Gordon system.

- There is no regular, time-independent configuration for complex scalar fields but one can construct harmonic time-dependence that produce time-independent metric.

- We adopt the following ansatz for boson stars in spherical symmetry in order to produce a static spacetime:

\[
\phi(t, r) = \phi_0(r) e^{-i\omega t}, \quad \beta = 0
\]

(39)

where the last condition comes from the demand of a static timelike Killing vector field.

- **Polar-Areal coordinates**

\[
K = K^r_r, \quad b = 1
\]

(40)

- Generalization of the usual Schwarzschild coordinates to *time-dependent*, spherically symmetric spacetimes. Easier to generate the initial data solution.
Appendix B: Boson Stars in Spherical Symmetry

- The line element
  \[ ds^2 = -\alpha^2 dt^2 + a^2 dr^2 + r^2 d\Omega^2. \] (41)

- The equations of motions are cast in a system of ODEs. It becomes an eigenvalue problem with eigenvalue \( \omega = \omega(\phi_0(0)) \)

\[
\begin{align*}
  a' &= \frac{1}{2} \left\{ \frac{a}{r} (1 - a^2) + 4\pi r a \left[ \phi^2 a^2 \left( m^2 + \frac{\omega^2}{\alpha^2} \right) + \Phi^2 \right] \right\} \tag{42} \\
  \alpha' &= \frac{\alpha}{2} \left\{ \frac{a^2 - 1}{r} + 4\pi r \left[ a^2 \phi^2 \left( \frac{\omega^2}{\alpha^2} - m^2 \right) + \Phi^2 \right] \right\} \tag{43} \\
  \phi' &= \Phi \tag{44} \\
  \Phi' &= - \left( 1 + a^2 - 4\pi r^2 a^2 m^2 \phi^2 \right) \frac{\Phi}{r} - \left( \frac{\omega^2}{\alpha^2} - m^2 \right) \phi a^2 \tag{45}
\end{align*}
\]
Appendix B: Boson Stars in Spherical Symmetry

- Field configuration and its aspect mass function for $\phi_0(0) = 0.05$. Its eigenvalue was "shot" to be $\omega = 1.1412862322$

- Note its exponentially decaying tail as opposed to the sharp edge ones for its fluids counterparts
Appendix B: Boson Stars in Spherical Symmetry

- The ADM mass as a function of the central density and the radius of the star as a function of ADM mass. Note their similarity to the fluid stars.