A Numerical Study of Boson Star Binaries

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General Motivation

- Why study compact binaries?
  - One of most promising sources of gravitational waves.
  - It is a good laboratory to study the phenomenology of strong gravitational fields.

- Why boson stars?
  - Matter similarities: Fluid stars and Boson stars have some similarity concerning the way they are modelled, e.g. both can be parametrized by their central density ρ_0 and have qualitatively similar plots of total mass vs ρ_0.
  - Then in the strong field regime for the compact binary system the dynamics may not depend sensitively on the details of the model.
  - Inspiral phases: Plunge and merge phase of the inspiral of compact objects is characterized by a strong dynamical gravitational field. In this regime gross features of fluid and boson stars' dynamics may be similar.
  - Since the details of the dynamics of the stars (e.g. shocks) tend not to be important gravitationally, boson star binaries may provide some insight into NS binaries.

Matter Model: Scalar Field

- Star-like solutions: A massive complex field is chosen as matter source because it is a simple type of matter that allows a star-like solution and because there will be no problems with shocks, low density regions, ultrarelativistic flows, etc in the evolution of this kind of matter as opposed to fluids.

- Static spacetimes: *Complex* scalar fields allow the construction of static spacetimes. The matter content is then described by $\Phi = \phi_1 + i\phi_2$, where ϕ_1 and ϕ_2 are real-valued.

- Equations of motion: Klein-Gordon equation:

$$\Box\phi_A - m^2\phi_A = 0, \qquad A = 1, 2. \tag{1}$$

- Hamiltonian Formulation: In terms of the conjugate momentum field Π_A:

$$\partial_t\phi_A = \frac{\alpha^2}{\sqrt{-g}}\Pi_A + \beta^i\partial_i\phi_A, \tag{2}$$

$$\partial_t\Pi_A = \partial_i(\beta^i\Pi_A) + \partial_i(\sqrt{-g}\gamma^{ij}\partial_j\phi_A) - \sqrt{-g}m^2\phi_A. \tag{3}$$

Conformally Flat Approximation (CFA)

- Motivation
  - Facts and assumptions:
    - Full 3D Einstein equations are very complex and computationally expensive to solve.
    - Gravitational radiation is small in most systems studied so far.
    - Heuristic assumption that the dynamical degrees of freedom of the gravitational fields, i.e. the gravitational radiation, play a small role in at least some phases of the strong field interaction of a merging binary.
  - An approximation candidate:
    - CFA effectively eliminates the two dynamical degrees of freedom, simplifies the equations and allows a fully constrained evolution.
    - CFA allows us to investigate the same kind of phenomena observed in the full relativistic case, such as the description of compact objects and the dynamics of their interaction; black hole formation; critical phenomena.

Conformally Flat Approximation (CFA)

- Formalism
  - Einstein field equations cast into 3+1/ADM form.
  - The CFA prescribes a conformally flat spatial metric at all times.
  - Introduce a flat metric f_{ij} as a base / background metric:

$$\gamma_{ij} = \psi^4 f_{ij}, \tag{4}$$

    where the conformal factor ψ is a positive scalar function describing the ratio between the scale of distance in the curved space and flat space ($f_{ij} \equiv \delta_{ij}$ in cartesian coordinates).
  - Maximum slicing condition is used to fix the time coordinate:

$$K^i_i = 0,$$
$$\partial_t K^i_i = 0. \tag{5}$$

  - In this approximation all of the geometric variables can be computed from the constraints as well as from a specific choice of coordinates.

Conformally Flat Approximation (CFA)

- Slicing Condition
  - Gives an elliptic equation for the lapse function α:

$$\nabla^2\alpha = -\frac{2}{\psi}\vec{\nabla}\psi \cdot \vec{\nabla}\alpha + \alpha\psi^4 \left(K_{ij}K^{ij} + 4\pi\left(\rho + S\right)\right). \tag{6}$$

- Hamiltonian Constraint
  - Gives an elliptic equation for the conformal factor ψ:

$$\nabla^2\psi = -\frac{\psi^5}{8}\left(K_{ij}K^{ij} + 16\pi\rho\right). \tag{7}$$

- Momentum Constraints
  - Given elliptic equations for the shift vector components β^i:

$$\begin{aligned}\nabla^2\beta^j &= -\frac{1}{3}\hat{\gamma}^{ij}\partial_i\left(\vec{\nabla}\cdot\vec{\beta}\right) + \alpha\psi^4 16\pi J^j - \partial_i\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[\hat{\gamma}^{ik}\partial_k\beta^j\right. \\ &\quad \left. + \hat{\gamma}^{jk}\partial_k\beta^i - \frac{2}{3}\hat{\gamma}^{ij}\left(\vec{\nabla}\cdot\vec{\beta}\right)\right].\end{aligned} \tag{8}$$

Conformally Flat Approximation (CFA)

- Note that $K_{ij}K^{ij}$ can also be expressed in terms of the flat operators. It ends up being expressed as flat derivatives of the shift vector:

$$K_{ij}K^{ij} = \frac{1}{2\alpha^2}\left(\hat{\gamma}_{kn}\hat{\gamma}^{ml}\hat{D}_m\beta^k\hat{D}_l\beta^n + \hat{D}_m\beta^l\hat{D}_l\beta^m - \frac{2}{3}\hat{D}_l\beta^l\hat{D}_k\beta^k\right). \tag{9}$$

Conformally Flat Approximation (CFA)

- Then the following set of functions completely characterize the geometry at each time slice:

$$\alpha = \alpha(t, \vec{r}), \quad \psi = \psi(t, \vec{r}), \quad \beta^i = \beta^i(t, \vec{r}), \tag{10}$$

  where $\vec{r}$ depends on the coordinate choice for the spatial hypersurface.

- The solution of the gravitational system under CFA and maximal slicing condition can be summarized as:
  - Specify initial conditions for the complex scalar field.
  - Solve the elliptic equations for the geometric quantities on the initial slice.
  - Update the matter field values to the next slice using their equation of motion.
  - For the new configuration of matter fields, re-solve the elliptic equations for the geometric variables and again allow the matter fields to react and evolve to the next slice and so on.

Conformally Flat Approximation (CFA)

- Discretization Scheme:

$$Lu - f = 0 \qquad \Rightarrow \qquad L^h u^h - f^h = 0. \tag{11}$$

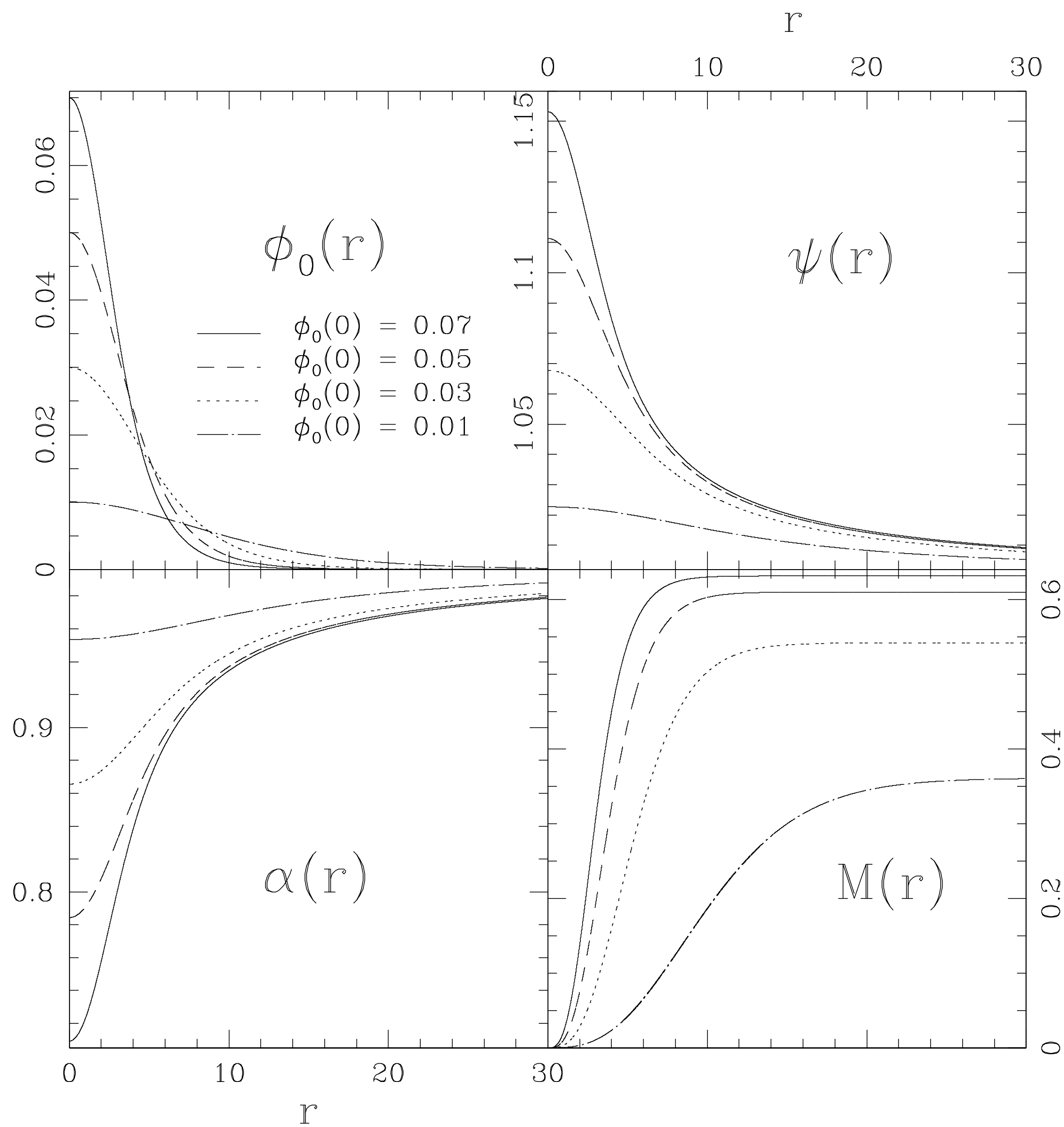
  - For hyperbolic operators L: second order accurate Crank-Nicholson scheme.
  - For elliptic operators L: second order accurate centred finite difference operators.
  - Dirichlet Boundary conditions applied.

- Numerical Techniques:

  - pointwise Newton-Gauss-Seidel (NGS) iterative technique was used to solve the finite difference equations originated from the hyperbolic set of equations.
  - Full Approximation Storage (FAS) multigrid algorithm was applied on the discrete version of the elliptic set of equations. NGS is used in this context as a smoother of the solution error.

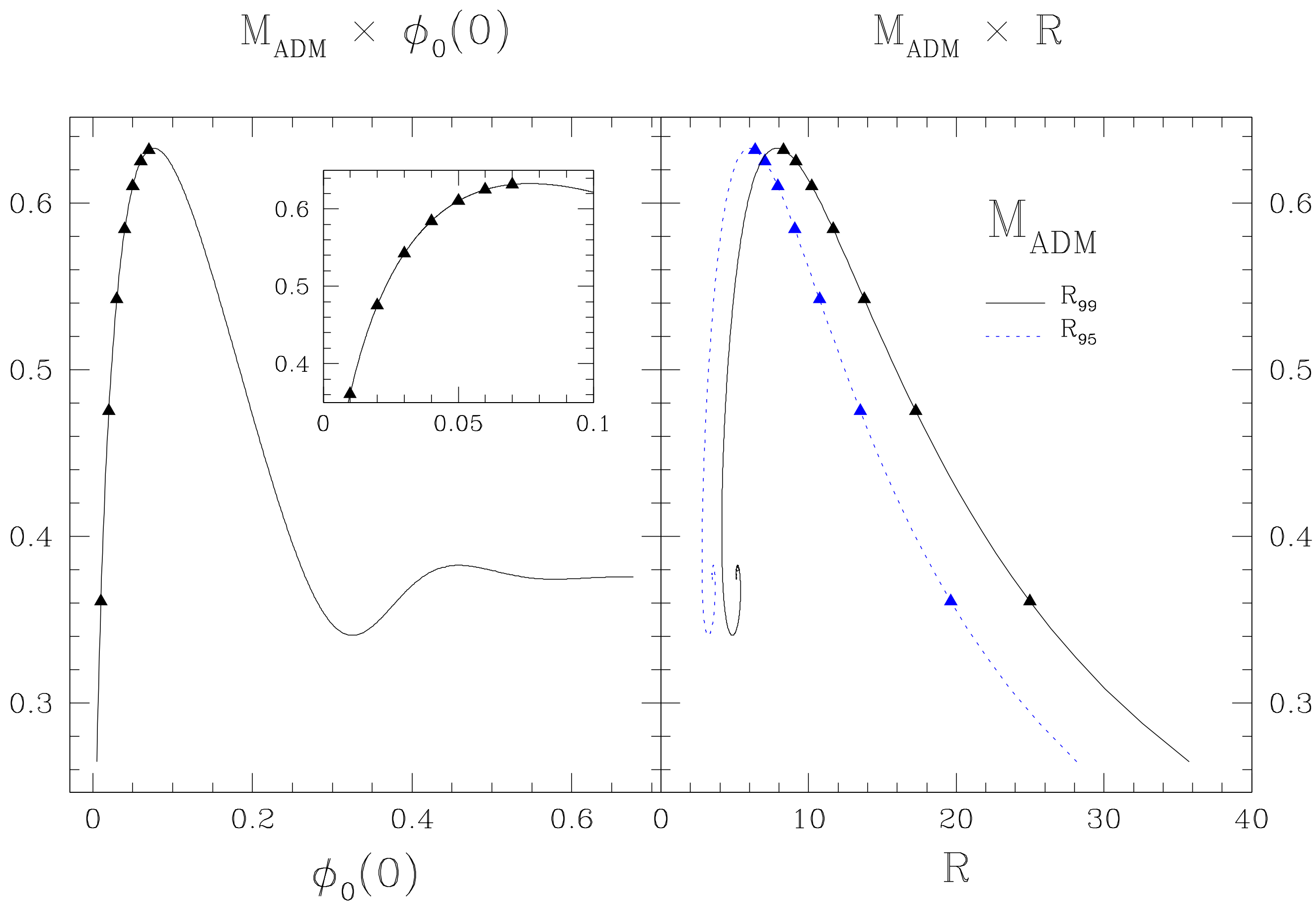
Initial Data

- Static spherically symmetric ansatz: $\phi(t,r) = \phi_0(r)e^{-i\omega t}$.



Initial Data

- Family of static spherically symmetric solutions:



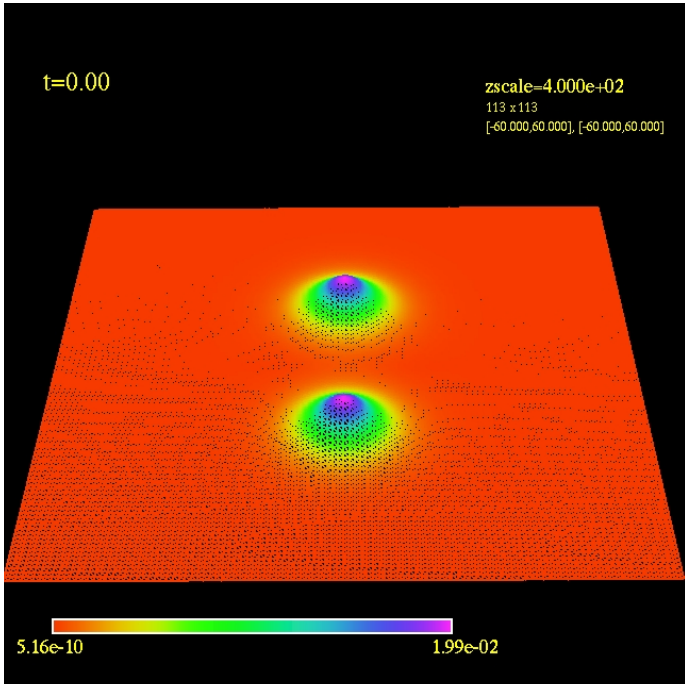
Evolution Results

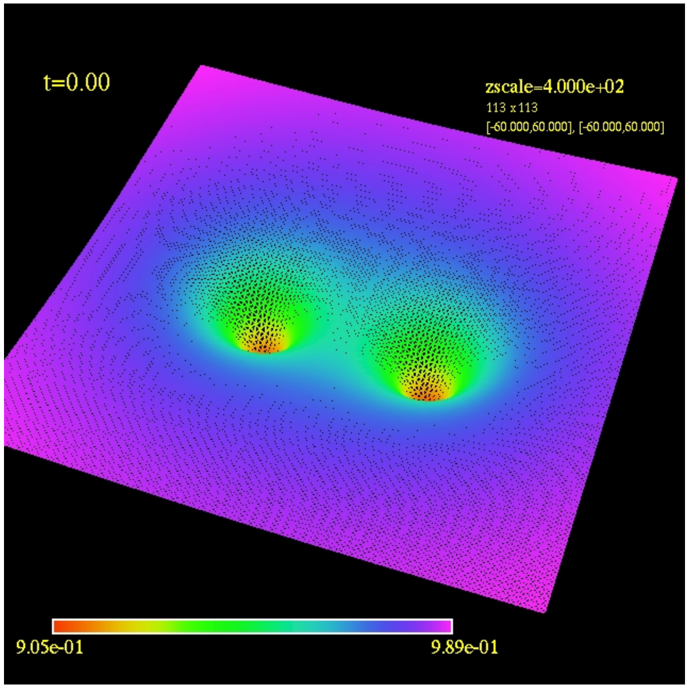
- Remarks: $\rho \sim |\phi(t,r)|^2 = \phi_0(r)^2$, and Planck units are adopted, i.e. $G = c = \hbar = 1$.

- Results summary:
  - Initial data:
    - Each boson star is modelled as a static, spherically symmetric solution of the Einstein-Klein-Gordon system.
    - They each have a central scalar field value of $\phi_0(0) = 0.02$, that corresponds to a boson star with radius of $R_{99} \simeq 17$ and ADM mass of $M_{ADM} \simeq 0.475$.
    - Each solution is then superposed and boosted in opposite directions (along x axis).
  - Evolution:
    - Orbital motion and interrupted orbits: The stars lie along the y axis with a coordinate separation between their centers of 40. Three distinct cases studied corresponding to different initial velocities:
      - $v_x = 0.09$: two orbital periods.
      - $v_x = 0.07$: rotating boson star as final merger.
      - $v_x = 0.05$: possible black-hole formation.
    - Head-on collision: Stars along x axis with coordinate separation of 50:
      - $v_x = 0.4$: solitonic behaviour.

Evolution Results

- Orbital Dynamics: 2 boson stars - $v_x = 0.09$.





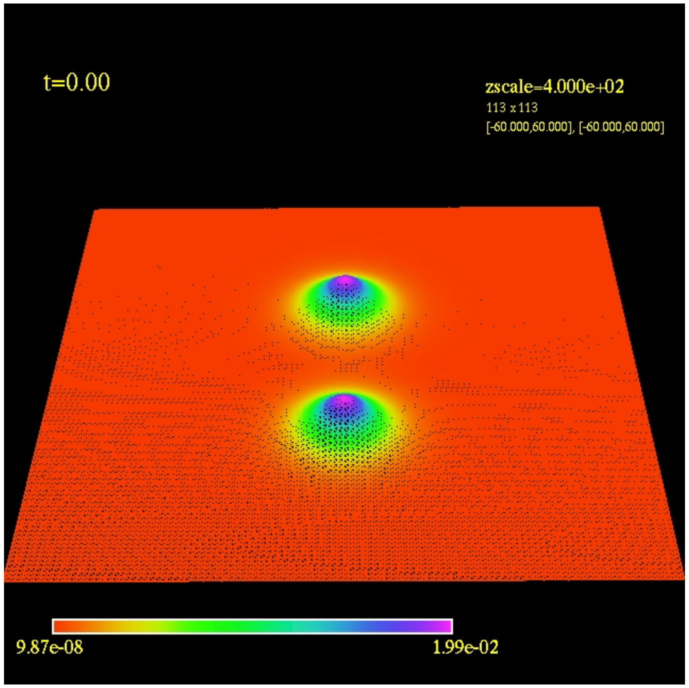
$Z = 0$ slice for $|\phi|$

$Z = 0$ slice for α

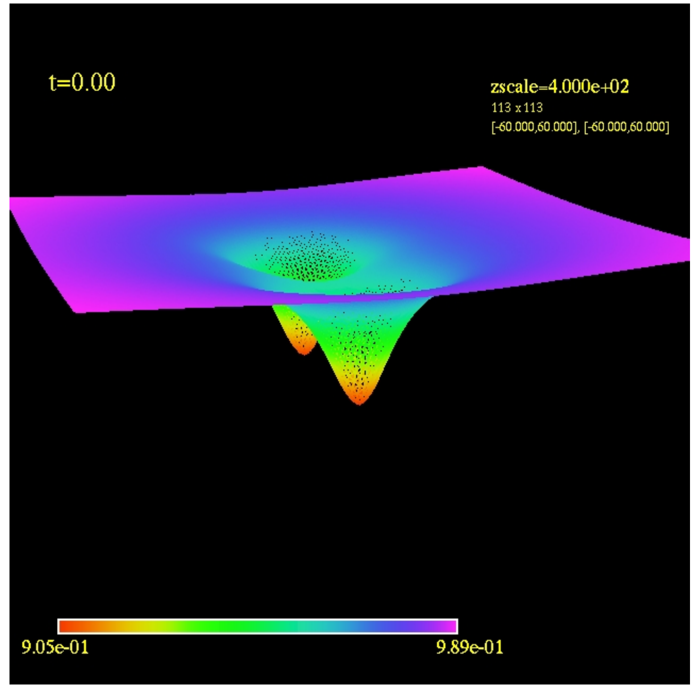
- $\phi_0 : 0.02$. Physical coordinate domain: 120 per edge. Physical time: $t = 4500$. Simulation parameters: Courant factor $\lambda = 0.4$; Grid size: 113^3; 2.4GHz Dual-Core AMD Opteron CPU time: 285 hours (12 days).

Evolution Results

- Orbital Dynamics: 2 boson stars - $v_x = 0.07$.



$Z = 0$ slice for $|\phi|$

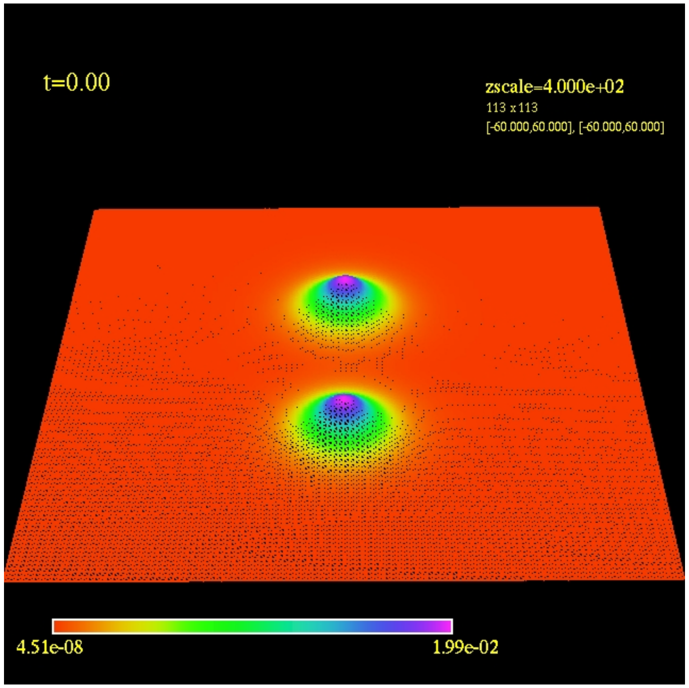


$Z = 0$ slice for α

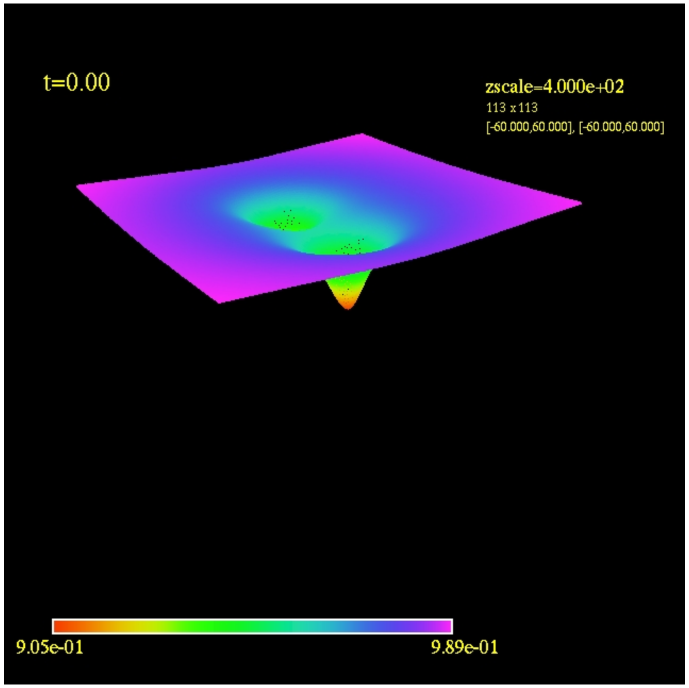
- $\phi_0 : 0.02$. Physical coordinate domain: 120 per edge. Physical time: $t = 2250$. Simulation parameters: Courant factor $\lambda = 0.4$; Grid size: 113^3; 2.4GHz Dual-Core AMD Opteron CPU time: 158 hours (6.5 days).

Evolution Results

- Orbital Dynamics: 2 boson stars - $v_x = 0.05$.



$Z = 0$ slice for $|\phi|$

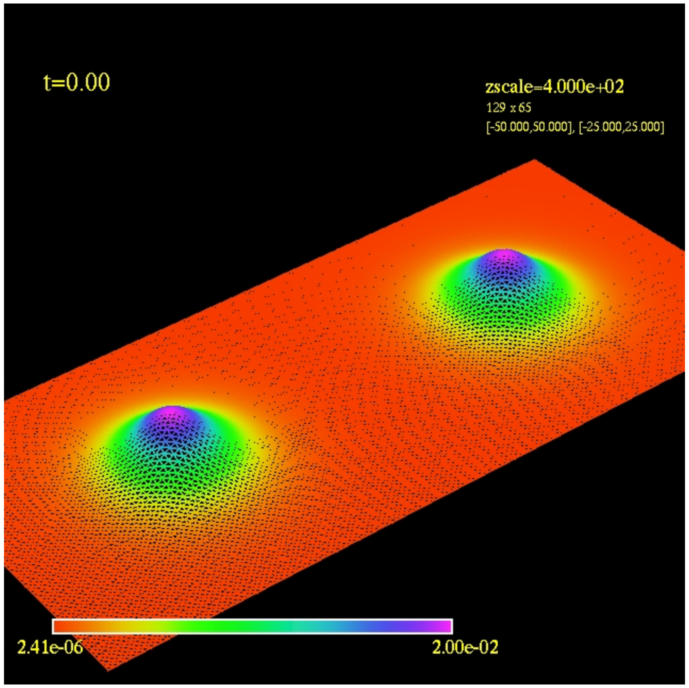


$Z = 0$ slice for α

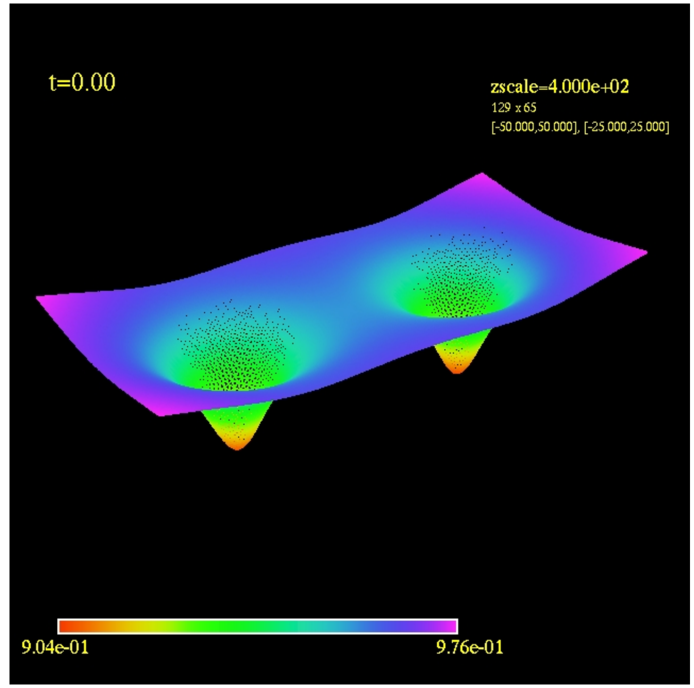
- $\phi_0 : 0.02$. Physical coordinate domain: 120 per edge. Physical time: $t = 1500$. Simulation parameters: Courant factor $\lambda = 0.4$; Grid size: 113^3; 2.4GHz Dual-Core AMD Opteron CPU time: 115 hours (4.5 days).

Evolution Results

- Head-on collision: 2 boson stars - $v_x = 0.4$.



$Z = 0$ slice for $|\phi|$



$Z = 0$ slice for α

- $\phi_0 : 0.02$. Physical coordinate domain: $[-50, 50, -25, 25, -25, 25]$. Total physical time: $t = 140$. Simulation parameters: $\lambda = 0.4$; Grid size: $[N_x, N_y, N_z] = [129, 65, 65]$;

Conclusion and Future Directions

- We were able to probe a few outcomes of the orbital dynamics of boson stars within the CFA.

- At least qualitatively, we observed a few characteristic phenomena of the fully relativistic case, such as orbital precession and scalar matter solitonic behaviour.

- These results are quite promising and suggest that, with enhancements such as the incorporation of AMR techniques and parallel execution capabilities, this code will be a powerful tool for investigating the strong gravity effects in the interaction of boson stars.

- We hope, in the future, to be able to calibrate the CFA fidelity to the fully general relativistic case and use this code to survey the parameter space of the orbital dynamics.

Appendix A - CFA equations of motion

- 3d Cartesian Coordinates

$$\begin{aligned} \partial_t \phi_A &= \frac{\alpha}{\psi^6} \Pi_A + \beta^i \partial_i \phi_A && (12) \\ \partial_t \Pi_A &= \partial_x \left(\beta^x \Pi_A + \alpha \psi^2 \partial_x \phi_A \right) + \partial_y \left(\beta^y \Pi_A + \alpha \psi^2 \partial_y \phi_A \right) && (13) \\ &+ \partial_z \left(\beta^z \Pi_A + \alpha \psi^2 \partial_z \phi_A \right) - \alpha \psi^6 \frac{dU(\phi_0^2)}{d\phi_0^2} \phi_A \end{aligned}$$

$$\frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial^2 \alpha}{\partial y^2} + \frac{\partial^2 \alpha}{\partial z^2} = -\frac{2}{\psi} \left[\frac{\partial \psi}{\partial x} \frac{\partial \alpha}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial \alpha}{\partial y} + \frac{\partial \psi}{\partial z} \frac{\partial \alpha}{\partial z} \right] + \alpha \psi^4 \left(K_{ij} K^{ij} + 4\pi \left(\rho + S \right) \right) \quad (14)$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -\frac{\psi^5}{8} \left(K_{ij} K^{ij} + 16\pi\rho \right) \quad (15)$$

Appendix A - CFA equations of motion

- x component of the shift vector in cartesian coordinates

$$\frac{\partial^2\beta^x}{\partial x^2}+\frac{\partial^2\beta^x}{\partial y^2}+\frac{\partial^2\beta^x}{\partial z^2} = -\frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial\beta^x}{\partial x}+\frac{\partial\beta^y}{\partial y}+\frac{\partial\beta^z}{\partial z}\right)+\alpha\psi^4\,16\pi J^x$$
$$-\frac{\partial}{\partial x}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[\frac{4}{3}\frac{\partial\beta^x}{\partial x}-\frac{2}{3}\left(\frac{\partial\beta^y}{\partial y}+\frac{\partial\beta^z}{\partial z}\right)\right]$$
$$-\frac{\partial}{\partial y}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[\frac{\partial\beta^x}{\partial y}+\frac{\partial\beta^y}{\partial x}\right]$$
$$-\frac{\partial}{\partial z}\left[ln\left(\frac{\psi^6}{\alpha}\right)\right]\left[\frac{\partial\beta^x}{\partial z}+\frac{\partial\beta^z}{\partial x}\right] \qquad (16)$$

- $K_{ij}K^{ij}$ in 3d cartesian coordinates

$$K_{ij}K^{ij} = \frac{1}{2\alpha^2}\Bigg[\left(\frac{\partial\beta^x}{\partial x}\right)^2+\left(\frac{\partial\beta^x}{\partial y}\right)^2+\left(\frac{\partial\beta^x}{\partial z}\right)^2+\left(\frac{\partial\beta^y}{\partial x}\right)^2+\left(\frac{\partial\beta^y}{\partial y}\right)^2+\left(\frac{\partial\beta^y}{\partial z}\right)^2$$
$$+\left(\frac{\partial\beta^z}{\partial x}\right)^2+\left(\frac{\partial\beta^z}{\partial y}\right)^2+\left(\frac{\partial\beta^z}{\partial z}\right)^2+\frac{\partial}{\partial x}\left(\beta^x\frac{\partial}{\partial x}+\beta^y\frac{\partial}{\partial y}+\beta^z\frac{\partial}{\partial z}\right)\beta^x$$
$$+\frac{\partial}{\partial y}\left(\beta^x\frac{\partial}{\partial x}+\beta^y\frac{\partial}{\partial y}+\beta^z\frac{\partial}{\partial z}\right)\beta^y+\frac{\partial}{\partial z}\left(\beta^x\frac{\partial}{\partial x}+\beta^y\frac{\partial}{\partial y}+\beta^z\frac{\partial}{\partial z}\right)\beta^z$$
$$-\frac{2}{3}\left(\frac{\partial\beta^x}{\partial x}+\frac{\partial\beta^x}{\partial x}+\frac{\partial\beta^x}{\partial x}\right)^2\Bigg] \qquad (17)$$

Appendix B: Boson Stars in Spherical Symmetry

- Spherically Symmetric Spacetime (SS):

$$ds^2 = \left(-\alpha^2 + a^2\beta^2\right) dt^2 + 2a^2\beta \, dtdr + a^2dr^2 + r^2b^2d\Omega^2 \,, \tag{18}$$

- Hamiltonian constraint:

$$-\frac{2}{arb}\left\{\left[\frac{(rb)'}{a}\right]' + \frac{1}{rb}\left[\left(\frac{rb}{a}(rb)'\right)' - a\right]\right\} + 4K^r{}_r K^\theta{}_\theta + 2K^\theta{}_\theta{}^2 = 8\pi\left[\frac{|\Phi|^2 + |\Pi|^2}{a^2} + m^2|\phi|^2\right] \tag{19}$$

- Momentum constraint:

$$K^\theta{}_\theta{}' + \frac{(rb)'}{rb}\left(K^\theta{}_\theta - K^r{}_r\right) = \frac{2\pi}{a}\left(\Pi^*\Phi + \Pi\Phi^*\right) \,. \tag{20}$$

where the auxiliary field variables were defined as:

$$\Phi \equiv \phi' \,, \tag{21}$$

$$\Pi \equiv \frac{a}{\alpha}\left(\dot\phi - \beta\phi'\right) \,, \tag{22}$$

Boson Stars in Spherical Symmetry

- Evolution equations

$$\dot{a} = -\alpha a K^r{}_r + (a\beta)' \tag{23}$$

$$\dot{b} = -\alpha b K^\theta{}_\theta + \frac{\beta}{r}(rb)' \,. \tag{24}$$

$$\dot{K^r{}_r} = \beta K^r{}_r{}' - \frac{1}{a}\left(\frac{\alpha'}{a}\right)' + \alpha\left\{-\frac{2}{arb}\left[\frac{(rb)'}{a}\right]' + KK^r{}_r - 4\pi\left[\frac{2|\Phi|^2}{a^2} + m^2|\phi|^2\right]\right\} \tag{25}$$

$$\dot{K^\theta{}_\theta} = \beta K^\theta{}_\theta{}' + \frac{\alpha}{(rb)^2} - \frac{1}{a(rb)^2}\left[\frac{\alpha rb}{a}(rb)'\right]' + \alpha\left(KK^\theta{}_\theta - 4\pi m^2|\phi|^2\right) \tag{26}$$

- Field evolution equations

$$\dot{\phi} = \frac{\alpha}{a}\Pi + \beta\Phi \tag{27}$$

$$\dot{\Phi} = \left(\beta\Phi + \frac{\alpha}{a}\Pi\right)' \tag{28}$$

$$\dot{\Pi} = \frac{1}{(rb)^2}\left[(rb)^2\left(\beta\Pi + \frac{\alpha}{a}\Phi\right)\right]' - \alpha a m^2\phi + 2\left[\alpha K^\theta{}_\theta - \beta\frac{(rb)'}{rb}\right]\Pi \tag{29}$$

Appendix B: Boson Stars in Spherical Symmetry

- Maximal-isotropic coordinates
  - Maximal slicing condition

$$K \equiv K^i_i = 0 \qquad \dot{K}(t,r) = 0 \tag{30}$$

  - Isotropic condition

$$a = b \equiv \psi(t,r)^2 \tag{31}$$

  - They fix the lapse and shift (equivalent of fixing the coordinate system)

$$\alpha'' + \frac{2}{r\psi^2}\frac{d}{dr^2}\left(r^2\psi^2\right)\alpha' + \left[4\pi\psi^4 m^2|\phi|^2 - 8\pi|\Pi|^2 - \frac{3}{2}\left(\psi^2 K^r{}_r\right)^2\right]\alpha = 0 \tag{32}$$

$$r\left(\frac{\beta}{r}\right)' = \frac{3}{2}\alpha K^r{}_r \tag{33}$$

  - Constraint equations

$$\frac{3}{\psi^5}\frac{d}{dr^3}\left(r^2\frac{d\psi}{dr}\right) + \frac{3}{16}K^r{}_r{}^2 = -\pi\left(\frac{|\Phi|^2+|\Pi|^2}{\psi^4} + m^2|\phi|^2\right) \tag{34}$$

$$K^r{}_r{}' + 3\frac{(r\psi^2)'}{r\psi^2}K^r{}_r = -\frac{4\pi}{\psi^2}\left(\Pi^*\Phi + \Pi\Phi^*\right) \tag{35}$$

Appendix B: Boson Stars in Spherical Symmetry

- Complex-scalar field evolution equations

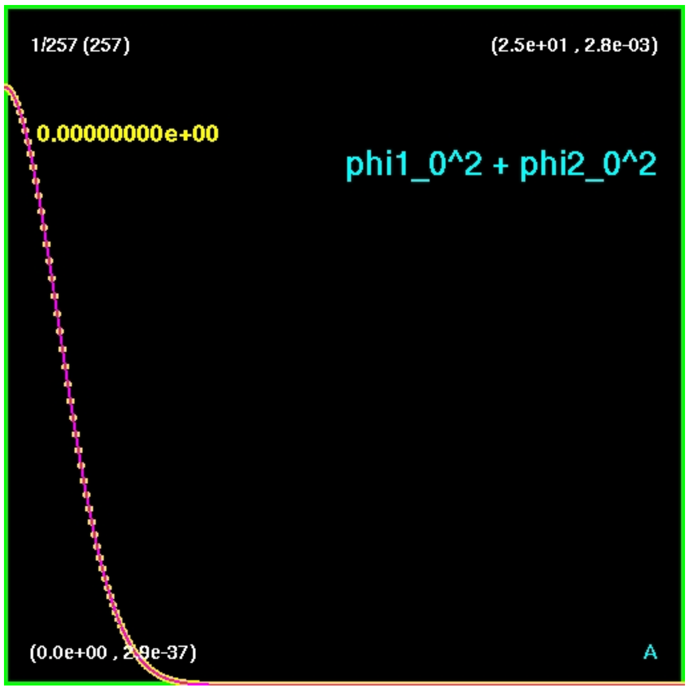
$$\dot{\phi} = \frac{\alpha}{\psi^2}\Pi + \beta\Phi \tag{36}$$

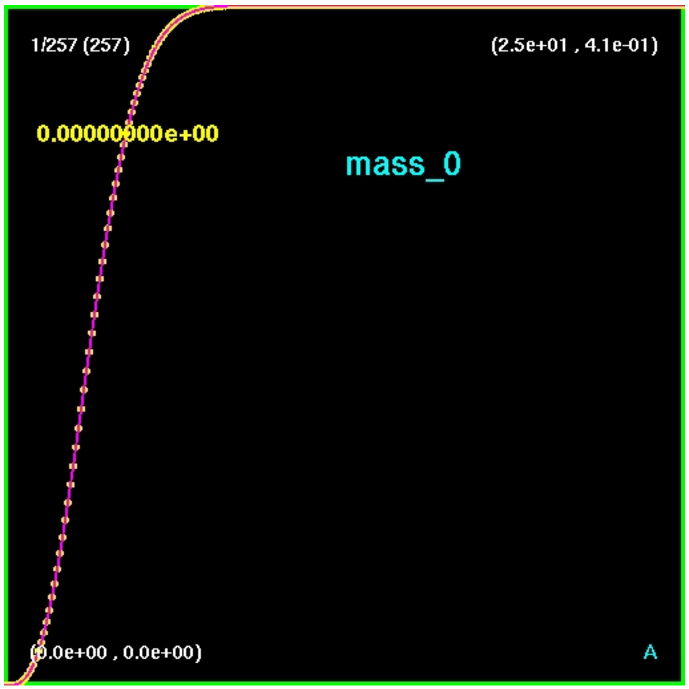
$$\dot{\Phi} = \left(\beta\Phi + \frac{\alpha}{\psi^2}\Pi\right)' \tag{37}$$

$$\dot{\Pi} = \frac{3}{\psi^4}\frac{d}{dr^3}\left[r^2\psi^4\left(\beta\Pi + \frac{\alpha}{\psi^2}\Phi\right)\right] - \alpha\psi^2 m^2\phi - \left[\alpha K^r{}_r + 2\beta\frac{(r\psi^2)'}{r\psi^2}\right]\Pi \tag{38}$$

Appendix B: Boson Stars in Spherical Symmetry

- These equations were coded using RNPL and tested for a gaussian pulse as initial data.





Appendix B: Boson Stars in Spherical Symmetry

- Initial Value Problem

- We are interested in generating *static* solutions of the Einstein- Klein-Gordon system

- There is no regular, time-independent configuration for complex scalar fields but one can construct harmonic time-dependence that produce time-independ ent metric

- We adopt the following ansatz for boson stars in spherical symmetry in order to produce a static spacetime:

$$\phi(t,r) = \phi_0(r)\, e^{-i\omega t}, \qquad \beta = 0 \tag{39}$$

  where the last condition comes from the demand of a static timelike Killing vector field.

- Polar-Areal coordinates

$$K = K^r{}_r \qquad b = 1 \tag{40}$$

- Generalization of the usual Schwarzschild coordinates to *time-dependent*, spherically symmetric spacetimes. Easier to generate the initial data solution

Appendix B: Boson Stars in Spherical Symmetry

- The line element

$$ds^2 = -\alpha^2 dt^2 + a^2 dr^2 + r^2 d\Omega^2 \,. \tag{41}$$

- The equations of motions are cast in a system of ODEs. It becomes an eigenvalue problem with eigenvalue $\omega = \omega(\phi_0(0))$

$$a' = \frac{1}{2}\left\{\frac{a}{r}\left(1-a^2\right) + 4\pi r a\left[\phi^2 a^2\left(m^2+\frac{\omega^2}{\alpha^2}\right)+\Phi^2\right]\right\} \tag{42}$$

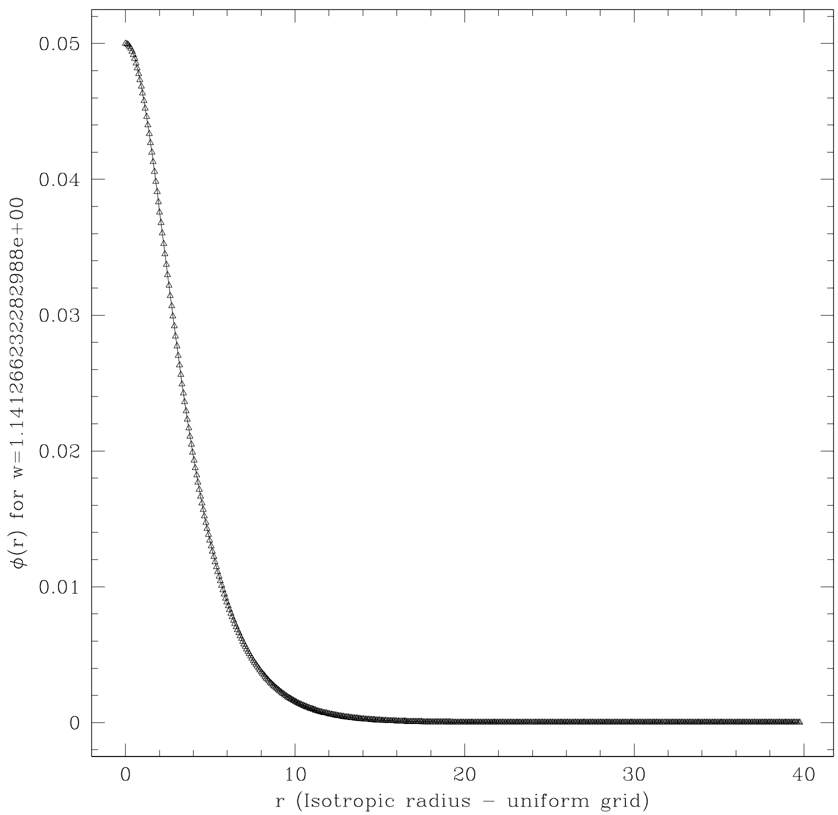
$$\alpha' = \frac{\alpha}{2}\left\{\frac{a^2-1}{r} + 4\pi r\left[a^2\phi^2\left(\frac{\omega^2}{\alpha^2}-m^2\right)+\Phi^2\right]\right\} \tag{43}$$

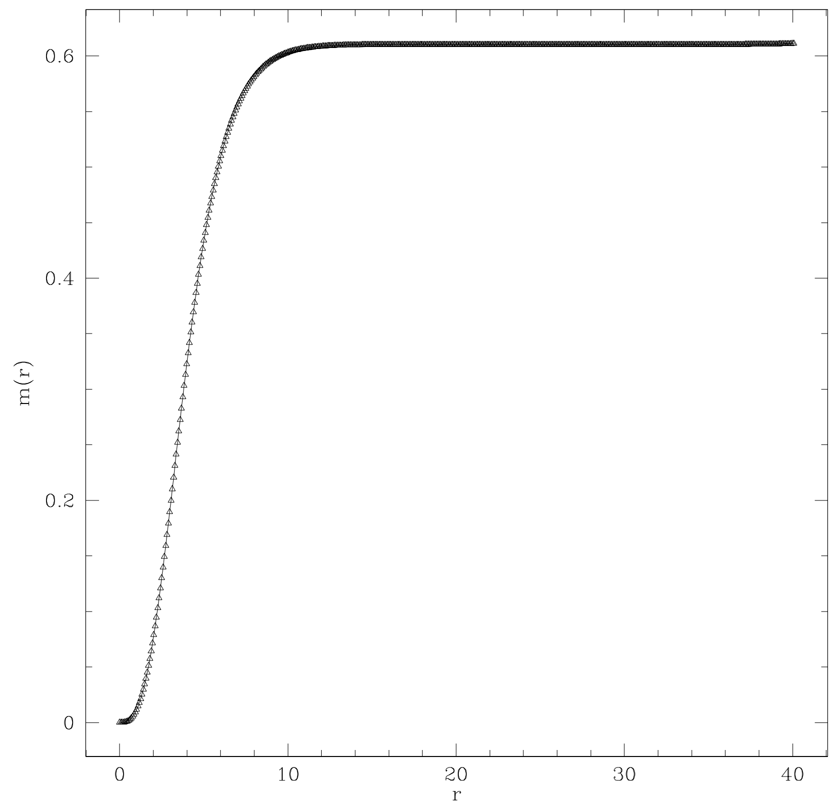
$$\phi' = \Phi \tag{44}$$

$$\Phi' = -\left(1+a^2-4\pi r^2 a^2 m^2 \phi^2\right)\frac{\Phi}{r} - \left(\frac{\omega^2}{\alpha^2}-m^2\right)\phi a^2 \tag{45}$$

Appendix B: Boson Stars in Spherical Symmetry

- Field configuration and its aspect mass function for $\phi_0(0) = 0.05$. Its eigenvalue was "shooted" to be $\omega = 1.1412862322$





- Note its exponentially decaying tail as opposed to the sharp edge ones for its fluids counterparts

Appendix B: Boson Stars in Spherical Symmetry

- The ADM mass as a function of the central density and the radius of the star as a function of ADM mass. Note their similarity to the fluid stars