STAT 406-01: Mathematical Statistics II

Problem Set 1

Assigned 2016 January 26
Due 2016 February 2

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems. Note that if using an outside source to do a calculation, you should use it as a reference for the method, and actually carry out the calculation yourself; it's not sufficient to quote the results of a calculation contained in an outside source.

1 Distributivity

Construct a truth table to show that $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$ (This should have eight rows corresponding to the possible combinations of truth and falsehood of A, B, and C,)

2 Extended Sum Rule

Use the product rule $P(A, B|I) = P(A|I)\, P(B|A, I)$ and the sum rule $P(A|I) + P(\overline{A}|I) = 1$, along with the rules of logic, to show that, in general, $P(A \vee B|I) = P(A|I) + P(B|I) - P(A \wedge B|I)$.

3 Independence

Suppose that $0 < P(B|I) < 1$.

(a) Use the rules of probability to write $P(A|I)$ as a weighted average $P(A|I) = a\, P(A|B, I) + b\, P(A|\overline{B}, I)$ where $a + b = 1$, being sure to specify the weights a and b.

(b) Show that $P(A|B, I) = P(A|I)$ implies $P(A|\overline{B}, I) = P(A|I)$.

4 Hogg 11.2.1

5 Hogg 11.2.2

6 Probability and Frequency

The purpose of this problem is to verify that the broader Bayesian definition of probability agrees with the classical long-term frequency definition when the latter can be defined. Suppose I represents a state of information that says that the probability of each of a set of propositions $\{A_i|i = 1, \ldots, n\}$ is p, and that the probability for a particular A_i is unaffected by knowledge about the truth or falsehood of any other set of $\{A_j\}$ not including A_i, i.e.,

$$P(A_i|I) = P(A_i|A_j, I) = P(A_i|A_j, A_k, I) = P(A_i|A_j, \overline{A_k}, I) = \cdots = p$$
$$i = 1, \ldots, n; j = 1, \ldots n, \text{ etc}; i \neq j, j \neq, k, i \neq k, \text{ etc.} \tag{6.1}$$

(a) Let $\{B_1, \ldots, B_n\}$ be a sequence of propositions in which each B_i is either A_i or $\{A_i\}$, with a total of k of the $\{B_i\}$ being the corresponding A_i and $n-k$ being the corresponding $\overline{A_i}$. (E.g., $B_1 = A_1$, $B_2 = \overline{A_2}$, $B_3 = \overline{A_3}$ would be one possible sequence for $n = 3$ in which $k = 1$.) Work out the probability $P(B_1, \ldots, B_n|I)$. Be sure to justify each step of your derivation using the rules of probability theory worked out in class.

(b) For any integer k, where $0 \leq k \leq n$, basic combinatorics tells us that there are ${}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$ different sequences of the form considered in part (a). Label these propositions $\{D_{k,n}^{(1)}, D_{k,n}^{(2)}, \ldots, D_{k,n}^{({}_nC_k)}\}$. E.g., $D_{1,3}^{(1)} = A_1 \wedge \overline{A_2} \wedge \overline{A_3}$, $D_{1,3}^{(2)} = \overline{A_1} \wedge A_2 \wedge \overline{A_3}$, and $D_{1,3}^{(3)} = \overline{A_1} \wedge \overline{A_2} \wedge A_3$. Use the result of part (a) to find $P(D_{k,n}^{(m)}|I)$.

(c) Let $E_{k,n} = D_{k,n}^{(1)} \vee D_{k,n}^{(2)} \vee \cdots \vee D_{k,n}^{({}_nC_k)}$ be the proposition that *any* k of the n propositions $\{A_i\}$ are true, and the rest are false. Use the restricted sum rule to work out $P(E_{k,n}|I)$.

(d) Define the random variable Y_n by

$$P\left(Y_n = \frac{k}{n}\middle| I\right) = P(E_{k,n}|I) \qquad k = 0, \ldots, n \tag{6.2}$$

For any $\epsilon > 0$, use Chebyshev's inequality to find an upper bound bound on $P(|Y_n - p| \geq \epsilon|I)$, and show that this bound shrinks to zero as $n \to \infty$. This means that for a long enough sequence of trials, we predict with arbitrarily small uncertainty that the measured frequency will be as close as we want to the probability assigned to each proposition.