Consider the function \( f(z) = e^{z^2} \)

1. Recalling that \((x + iy)^2 = (x^2 - y^2) + i2xy\), write \( f(x + iy) = \rho(x, y)e^{i\phi(x, y)} \) where \( \rho(x, y) \) and \( \phi(x, y) \) are real functions of \( x \) and \( y \).

Since \( z^2 = (x + iy)^2 = (x^2 - y^2) + i2xy \) we have

\[
f(x + iy) = e^{(x+iy)^2} = e^{(x^2 - y^2) + i2xy} = e^{x^2 - y^2}e^{i2xy}
\]

\[
\rho(x, y) = e^{x^2 - y^2} \quad \phi(x, y) = 2xy
\]

2. Use the Euler relation \( e^{i\alpha} = \cos \alpha + i\sin \alpha \) and the results of part 1 to write \( f(x + iy) = u(x, y) + iv(x, y) \), where \( u(x, y) \) and \( v(x, y) \) are real functions of \( x \) and \( y \).

The Euler relation tells us \( e^{i2xy} = \cos 2xy + i\sin 2xy \), so

\[
f(x + iy)e^{x^2 - y^2}e^{i2xy} = e^{x^2 - y^2}(\cos 2xy + i\sin 2xy) = e^{x^2 - y^2}\cos 2xy + ie^{x^2 - y^2}\sin 2xy
\]

\[
u(x, y) = e^{x^2 - y^2}\cos 2xy \quad v(x, y) = e^{x^2 - y^2}\sin 2xy
\]
3. Take the partial derivatives of the \( u(x, y) \) and \( v(x, y) \) you found in part 2.

The partial derivatives of \( u(x, y) = e^{x^2 - y^2} \cos 2xy \) and \( e^{x^2 - y^2} \sin 2xy \) are

\[
\frac{\partial u}{\partial x} = (2xe^{x^2 - y^2}) \cos 2xy + e^{x^2 - y^2}(-2y \sin 2xy) = 2e^{x^2 - y^2}(x \cos 2xy - y \sin 2xy) \\
\frac{\partial u}{\partial y} = (-2ye^{x^2 - y^2}) \cos 2xy + e^{x^2 - y^2}(-2x \sin 2xy) = -2e^{x^2 - y^2}(y \cos 2xy + x \sin 2xy) \\
\frac{\partial v}{\partial x} = (2xe^{x^2 - y^2}) \sin 2xy + e^{x^2 - y^2}(2y \cos 2xy) = 2e^{x^2 - y^2}(y \cos 2xy + x \sin 2xy) \\
\frac{\partial v}{\partial y} = (-2ye^{x^2 - y^2}) \sin 2xy + e^{x^2 - y^2}(2x \cos 2xy) = 2e^{x^2 - y^2}(x \cos 2xy - y \sin 2xy)
\]

4. Use the results of part 3 to show \( f(z) = e^z \) is analytic everywhere.

Since \( \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \) and \( \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \), the Cauchy-Riemann equations are satisfied for all \( x \) and \( y \). This means \( f(z) \) is differentiable everywhere, which means it’s analytic everywhere.