Consider a continuous random variable $X$ with the uniform probability density function

$$f(x) = \begin{cases} \frac{1}{B-A} & A < x < B \\ 0 & \text{otherwise} \end{cases}$$

a. Verify that $f(x)$ is normalized, i.e., that

$$\int_{-\infty}^{\infty} f(x) \, dx = 1$$

The integral is

$$\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{A} f(x) \, dx + \int_{A}^{B} f(x) \, dx + \int_{B}^{\infty} f(x) \, dx = 0 + \int_{A}^{B} \frac{1}{B-A} \, dx + 0$$

$$= 0 + \frac{x}{B-A} \bigg|_{A}^{B} + 0 = \frac{B - A}{B - A} = 1$$

so $f(x)$ is indeed normalized.
b. Sketch the graph of $f(x)$. Label the axes.

![Graph of $f(x)$ with labels on axes and tickmarks]

(There are other possible choices of which tickmarks get which labels, but I wanted the 0 values to be clearly visible.) Note that the pdf is discontinuous, which is fine. Note also that it doesn’t really matter what value the pdf $f(x)$ takes on at those points ($x = A$ and $x = B$), since $f(x)$ always gets put under an integral to convert it into a probability.

c. Find the cumulative distribution $F(x)$.

The cdf is the probability

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y) \, dy$$

Note that we have to call the integration variable $y$ rather than $x$ because $x$ is the upper limit of the integral. If you have a definite integral, the integration variable should never appear in the limits of integration, nor anywhere outside the integral.

Because $f(x)$ has a different form for $x \leq A$, for $A \leq x \leq B$, and for $B \leq x$, the results of the integral will be different depending on where $x$ lies:

If $x \leq A$, \[ F(x) = \int_{-\infty}^{x} f(y) \, dy = 0 \]

because the integrand is zero over the whole range of integration.

If $A \leq x \leq B$, \[ F(x) = \int_{-\infty}^{A} f(y) \, dy + \int_{A}^{x} f(y) \, dy = 0 + \left. \frac{y}{B-A} \right|_{A}^{x} = \frac{x-A}{B-A} \]
Finally,

\[ F(x) = \int_{-\infty}^{A} f(y) \, dy + \int_{A}^{B} f(y) \, dy + \int_{B}^{x} f(y) \, dy = 0 + 1 + 0 = 1 \]

Putting it all together,

\[ F(x) = \begin{cases} 
0 & x \leq A \\
\frac{x-A}{B-A} & A \leq x \leq B \\
1 & B \leq x 
\end{cases} \]

Alternate solution using indefinite integrals:

Note that you can also do this by noting that since \( F'(x) = f(x), \)

\[ F(x) = \int f(x) \, dx \]

where now this is an indefinite integral. Then we have to take the antiderivative of the form of \( f(x) \) in each interval:

- If \( x \leq A, \) \( F(x) = \int 0 \, dx = C_1 \)
- If \( A \leq x \leq B, \) \( F(x) = \int \frac{1}{B-A} \, dx = \frac{x}{B-A} + C_2 \)
- If \( B \leq x, \) \( F(x) = \int 0 \, dx = C_3 \)

Because these are indefinite integrals, we have to include an arbitrary constant (\( C_1, C_2 \) and \( C_3 \), respectively), which is in general different for each integral. Then we need to find the values of these constants which ensure that \( F(-\infty) = 0 \) and that \( F(x) \) is continuous, which it must be for a continuous random variable. (We can then check that \( F(\infty) = 1 \), which must be the case if the pdf \( f(x) \) was properly normalized, and we didn’t make any mistakes.) We find \( C_1 \) from

\[ F(-\infty) = C_1 = 0 \]

so that

\[ F(x) = 0 \]

if \( x \leq A \)

and then find \( C_2 \) from continuity at \( x = A \)

\[ F(A) = 0 = \frac{A}{B-A} + C_2 \]

so that

\[ C_2 = -\frac{A}{B-A} \]

and

\[ F(x) = \frac{x}{B-A} - \frac{A}{B-A} \]

if \( A \leq x \leq B \).
We then find $C_3$ from continuity at $x = B$

$$F(B) = \frac{B}{B-A} - \frac{A}{B-A} = 1 = C_3$$

and thus

$$\text{If } B \leq x, \quad F(x) = 1$$

Finally, we can then see that $F(\infty) = 1$, so everything is consistent.

In the end the indefinite integral approach gives the right answer (of course) if you’re careful about the integration constants. Personally, I find the approach with definite integrals to be easier, since the matching happens automatically.

d. Sketch the graph of $F(x)$. Label the axes.

![Graph of F(x)](image)

Notice that the graph is continuous, as it must be for a continuous random variable.

e. Calculate the expected value $E(X)$ in terms of $A$ and $B$.

$$E(X) = \int_{-\infty}^{\infty} x f(x) \, dx = \int_{A}^{B} \frac{x}{B-A} \, dx = \frac{1}{B-A} \left[ \frac{x^2}{2} \right]_{A}^{B} = \frac{B^2 - A^2}{2(B-A)}$$

$$= \frac{(B + A)(B - A)}{2(B - A)} = \frac{A + B}{2}$$
f. Calculate the variance \( V(X) \) in terms of \( A \) and \( B \).

The easiest way to do this is to calculate

\[
E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) \, dx = \int_{A}^{B} \frac{x^2}{B-A} \, dx = \frac{1}{B-A} \left. \frac{x^3}{3} \right|_{A}^{B} = \frac{B^3 - A^3}{3(B-A)}
\]

\[
= \frac{(B^2 + AB + A^2)(B-A)}{3(B-A)} = \frac{B^2 + AB + A^2}{3}
\]

And then

\[
V(X) = E(X^2) - (E(X))^2 = \frac{B^2 + AB + A^2}{3} - \left( \frac{A+B}{2} \right)^2 = \frac{B^2 + AB + A^2}{3} - \frac{A^2 + 2AB + B^2}{4}
\]

\[
= \frac{4B^2 + 4AB + 4A^2 - 3A^2 - 6AB - 3B^2}{12} = \frac{B^2 - 2AB + A^2}{12} = \frac{(B - A)^2}{12}
\]