We’ve seen how to convert the complex number $z = x + iy = re^{i\theta}$ between Cartesian coördinates $(x, y)$ and polar coördinates $(r, \theta)$ using the transformations

$$x = r \cos \theta \quad (1a)$$
$$y = r \sin \theta \quad (1b)$$
$$r = \sqrt{x^2 + y^2} \quad (2a)$$
$$\theta = \text{atan2}(y, x) \quad (2b)$$

together with a table of trig functions of multiples of $\pi/4$ and $\pi/6$. Rather than memorizing or reconstructing that table of trig functions, one can instead remember the following two triangles (shown with their angles labelled in degrees and then in radians):

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Examples

1. **Convert** \( z = -\sqrt{3} + i \) **into polar form.**

First we read off from \( z = x + iy = -\sqrt{3} + i \) the Cartesian coördinates

\[
\begin{align*}
    x &= -\sqrt{3} \\
    y &= 1 .
\end{align*}
\]

(3a) (3b)

This point lies in the second quadrant:

The sides \( \sqrt{3} \) and 1 fit into our \( 30^\circ-60^\circ-90^\circ \) triangle, which we blow up in the right-hand side of the diagram. The modulus is \( r = |z| = \sqrt{x^2 + y^2} = \sqrt{3 + 1} = 2 \), which can also be seen from the fact that the hypotenuse of the triangle is 2. We see that the angle \( \theta \) is between \( \pi/2 \) (90°) and \( \pi \) (180°). Since the small angle of the triangle is \( \pi/6 \), we must have

\[
\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6} ,
\]

(4)

i.e.,

\[
z = -\sqrt{3} + i = 2e^{i5\pi/6} .
\]

(5)

Note that these triangles tell us that

\[
\begin{align*}
    \cos \theta &= \frac{x}{r} = \cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} \\
    \sin \theta &= \frac{y}{r} = \sin \frac{5\pi}{6} = \frac{1}{2} .
\end{align*}
\]

(6a) (6b)

Note also that we would have got the wrong answer if we’d taken

\[
\tan^{-1} \left( \frac{y}{x} \right) = \tan^{-1} \left( \frac{1}{-\sqrt{3}} \right) = \tan^{-1} \left( -\frac{1}{\sqrt{3}} \right) = -\frac{\pi}{6} \neq \theta .
\]

(7)
2. Convert \( z = 4e^{-i3\pi/4} \) into Cartesian form.

First we read off from \( z = re^{i\theta} = 4e^{-i3\pi/4} \) the polar coördinates

\[
\begin{align*}
    r &= 4 \\
    \theta &= -\frac{3\pi}{4}.
\end{align*}
\]

Since \( \theta = -3\pi/4 \) is between \(-\pi\) \((-180^\circ)\) and \(-\pi/2\) \((-90^\circ)\), the point lies in the third quadrant: Looking at the diagram and using \( \pi - 3\pi/4 = \pi/4 \), we see that the angles in the triangle are \(45^\circ\), which gives us the isosceles right triangle blown up on the right. We’ve scaled up the triangle by a factor of \(2\sqrt{2}\) so that the hypotenuse is \(r = 4\), from which we can read off

\[
\begin{align*}
x &= -2\sqrt{2} \\
y &= -2\sqrt{2},
\end{align*}
\]

i.e.,

\[
z = 4e^{-i3\pi/4} = -2\sqrt{2} - i2\sqrt{2}.
\]

Note that these triangles tell us that

\[
\begin{align*}
\cos \theta &= \frac{x}{r} = \cos \left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} \\
\sin \theta &= \frac{y}{r} = \sin \left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}.
\end{align*}
\]