Consider a continuous random variable with the uniform probability density function

\[
f(x) = \begin{cases} 
\frac{1}{B-A} & A < x < B \\
0 & \text{otherwise} 
\end{cases}
\]

a. Verify that \( f(x) \) is normalized, i.e., that

\[
\int_{-\infty}^{\infty} f(x) \, dx = 1
\]

The integral is

\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{A} f(x) \, dx + \int_{A}^{B} f(x) \, dx + \int_{B}^{\infty} f(x) \, dx = 0 + \int_{A}^{B} \frac{1}{B-A} \, dx + 0
\]

\[
= 0 + \left. \frac{x}{B-A} \right|_{A}^{B} + 0 = \frac{B-A}{B-A} = 1
\]

so \( f(x) \) is indeed normalized.
b. Sketch the graph of \( f(x) \). Label the axes.

(There are other possible choices of which tickmarks get which labels, but I wanted the 0 values to be clearly visible.) Note that the pdf is discontinuous, which is fine. Note also that it doesn’t really matter what value the pdf \( f(x) \) takes on at those points \( (x = A \) and \( x = B) \), since \( f(x) \) always gets put under an integral to convert it into a probability.

c. Find the cumulative distribution \( F(x) \).

The cdf is the probability

\[
F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y) \, dy
\]

Note that we have to call the integration variable \( y \) rather than \( x \) because \( x \) is the upper limit of the integral. If you have a definite integral, the integration variable should never appear in the limits of integration, nor anywhere outside the integral.

Because \( f(x) \) has a different form for \( x \leq A \), for \( A \leq x \leq B \), and for \( B \leq x \), the results of the integral will be different depending on where \( x \) lies:

If \( x \leq A \), \n\[
F(x) = \int_{-\infty}^{x} f(y) \, dy = 0
\]

because the integrand is zero over the whole range of integration.

If \( A \leq x \leq B \), \n\[
F(x) = \int_{-\infty}^{A} f(y) \, dy + \int_{A}^{x} f(y) \, dy = 0 + \frac{y}{B - A}\bigg|_{A}^{x} = \frac{x - A}{B - A}
\]
Finally,

If $B \leq x$, \[ F(x) = \int_{-\infty}^{A} f(y) \, dy + \int_{A}^{B} f(y) \, dy + \int_{B}^{x} f(y) \, dy = 0 + 1 + 0 = 1 \]

Putting it all together,

\[ F(x) = \begin{cases} 0 & x \leq A \\ \frac{x-A}{B-A} & A \leq x \leq B \\ 1 & B \leq x \end{cases} \]

Alternate solution using indefinite integrals:

Note that you can also do this by noting that since $F'(x) = f(x)$,

\[ F(x) = \int f(x) \, dx \]

where now this is an indefinite integral. Then we have to take the antiderivative of the form of $f(x)$ in each interval:

If $x \leq A$, \[ F(x) = \int 0 \, dx = C_1 \]

If $A \leq x \leq B$, \[ F(x) = \int \frac{1}{B-A} \, dx = \frac{x}{B-A} + C_2 \]

If $B \leq x$, \[ F(x) = \int 0 \, dx = C_3 \]

Because these are indefinite integrals, we have to include an arbitrary constant ($C_1$, $C_2$ and $C_3$, respectively), which is in general different for each integral. Then we need to find the values of these constants which ensure that $F(-\infty) = 0$ and that $F(x)$ is continuous, which it must be for a continuous random variable. (We can then check that $F(\infty) = 1$, which must be the case if the pdf $f(x)$ was properly normalized, and we didn’t make any mistakes.) We find $C_1$ from

\[ F(-\infty) = C_1 = 0 \]

so that

If $x \leq A$, \[ F(x) = 0 \]

and then find $C_2$ from continuity at $x = A$

\[ F(A) = 0 = \frac{A}{B-A} + C_2 \]

so that

\[ C_2 = -\frac{A}{B-A} \]

and

if $A \leq x \leq B$, \[ F(x) = \frac{x}{B-A} - \frac{A}{B-A} \]
We then find $C_3$ from continuity at $x = B$

$$F(B) = \frac{B}{B - A} - \frac{A}{B - A} = 1 = C_3$$

and thus

$$\text{If } B \leq x, \quad F(x) = 1$$

Finally, we can then see that $F(\infty) = 1$, so everything is consistent.

In the end the indefinite integral approach gives the right answer (of course) if you’re careful about the integration constants. Personally, I find the approach with definite integrals to be easier, since the matching happens automatically.

d. Sketch the graph of $F(x)$. Label the axes.

![Graph of $F(x)$]

Notice that the graph is continuous, as it must be for a continuous random variable.

e. Calculate the expected value $E(X)$ in terms of $A$ and $B$.

$$E(X) = \int_{-\infty}^{\infty} x f(x) \, dx = \int_{A}^{B} x \frac{1}{B - A} \, dx = \frac{1}{B - A} \left[ \frac{x^2}{2} \right]_{A}^{B} = \frac{B^2 - A^2}{2(B - A)}$$

$$= \frac{(B + A)(B - A)}{2(B - A)} = \frac{A + B}{2}$$
f. Calculate the variance $V(X)$ in terms of $A$ and $B$.

The easiest way to do this is to calculate

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) \, dx = \int_{A}^{B} \frac{x^2}{B-A} \, dx = \frac{1}{B-A} \left. \frac{x^3}{3} \right|_{A}^{B} = \frac{B^3 - A^3}{3(B-A)}$$

$$= \frac{(B^2 + AB + A^2)(B - A)}{3(B-A)} = \frac{B^2 + AB + A^2}{3}$$

And then

$$V(X) = E(X^2) - (E(X))^2 = \frac{B^2 + AB + A^2}{3} - \left( \frac{A + B}{2} \right)^2 = \frac{B^2 + AB + A^2}{3} - \frac{A^2 + 2AB + B^2}{4}$$

$$= \frac{4B^2 + 4AB + 4A^2 - 3A^2 - 6AB - 3B^2}{12} = \frac{B^2 - 2AB + A^2}{12} = \frac{(B - A)^2}{12}$$