# 1060-710
# Mathematical and Statistical Methods for Astrophysics

Problem Set 8

Assigned 2010 November 2
Due 2010 November 11

**Show your work on all problems!** Be sure to give credit to any collaborators, or outside sources used in solving the problems. Note that if using an outside source to do a calculation, you should use it as a reference for the method, and actually carry out the calculation yourself; it's not sufficient to quote the results of a calculation contained in an outside source.

## 0 Useful Integrals

The following integrals are likely to be useful in this problem set:

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}} \tag{0.1a}$$

$$\int_{-\infty}^{\infty} x\,e^{-\alpha x^2}\,dx = 0 \tag{0.1b}$$

$$\int_{-\infty}^{\infty} x^2\,e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}} \tag{0.1c}$$

$$\int_{-\infty}^{\infty} x^3\,e^{-\alpha x^2}\,dx = 0 \tag{0.1d}$$

$$\int_{-\infty}^{\infty} x^4\,e^{-\alpha x^2}\,dx = \frac{3}{4\alpha^2}\sqrt{\frac{\pi}{\alpha}} \tag{0.1e}$$

(Extra credit: show that each of these is true.)

## 1 Gaussian Random Data

Consider a particular Fourier component $\widehat{H}_k$ of a random data series, and, writing[1] $\widehat{H}_k = \Xi + i\mathrm{H}$, let its real and imaginary parts be random variables whose joint pdf is

$$f(\widehat{h}_k) = f(\xi,\eta) = \frac{\exp(-\xi^2/S_k)}{\sqrt{\pi S_k}}\frac{\exp(-\eta^2/S_k)}{\sqrt{\pi S_k}} = \frac{\exp(-|\widehat{h}_k|^2/S_k)}{\pi S_k} \tag{1.1}$$

so that the expectation value of any function of $\widehat{H}_k$ is

$$\left\langle g(\widehat{H}_k)\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi+i\eta)\,f(\xi,\eta)\,d\xi\,d\eta \tag{1.2}$$

[1] The H with no subscript is a capital $\eta$, not a capital $h$.

a) Show that

$$\left\langle \left|\widehat{H}_k\right|^2 \right\rangle = S_k \tag{1.3}$$

b) Show that

$$\left\langle \left|\widehat{H}_k\right|^4 \right\rangle = 2S_k^2 \tag{1.4}$$

(This means that expected standard deviation of the periodogram $P_k = \frac{\delta t}{N}\left|\widehat{H}_k\right|^2$ is equal to its expectation value mean.)

c) Consider the case where the original data stream is complex, so that the real and imaginary parts of $\widehat{H}_k$ for $k = -N/2, \ldots, N/2-1$ are independent random variables. Write the joint pdf

$$f(\{\widehat{h}_k\}) = \prod_{k=-N/2}^{N/2-1} f(\widehat{h}_k) \tag{1.5}$$

in the form

$$f(\{\widehat{h}_k\}) = \mathcal{A}\exp(L(\{\widehat{h}_k\})) \tag{1.6}$$

where $\mathcal{A}$ is a normalization constant independent of the Fourier components, and $L(\{\widehat{h}_k\})$ is some explicit expression in terms of the $\{\widehat{h}_k\}$ and $\{S_k\}$.

d) Use the identifications

$$\widehat{H}_k \sim (\delta t)^{-1}\ \widetilde{H}(f_k) \tag{1.7a}$$

$$\left\langle \left|\widehat{H}_k\right|^2 \right\rangle \sim \frac{N}{\delta t} S_h(f_k) \tag{1.7b}$$

to write $L(\{\widehat{h}_k\}) \sim L[\widetilde{h}]$ in the limit that $\delta t$ and $\delta f$ go to zero as a definite integral over $f$. (There should be no reference to anything discrete in your answer.) You have thereby worked out the pdf

$$f[\widetilde{h}] = \mathcal{A}\exp(L[\widetilde{h}]) \tag{1.8}$$

up to an overall normalization $\mathcal{A}$.

e) Extra credit: repeat the process for the case where $\{\widehat{H}_k\}$ are the Fourier components associated with a random real time series; you may find it useful to make reference to the one-sided PSD $S_h^{\text{1-sided}}(f)$ defined in class. (See the lecture notes on Fourier methods.)

# 2 Monte Carlo Experiment

Implement and perform your designated monte carlo experiment. Submit a writeup of the method, results and conclusions, along with a copy of the source code used.