1060-710
Mathematical and Statistical Methods for Astrophysics

Problem Set 7

Assigned 2010 October 26
Due 2010 November 2

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems. Note that if using an outside source to do a calculation, you should use it as a reference for the method, and actually carry out the calculation yourself; it's not sufficient to quote the results of a calculation contained in an outside source.

1 Bayes Factor

Consider measurements $\{y_i\}$ taken at times $\{t_i\}$, which are assumed differ from values $\{\mu_i\}$ predicted by the "correct" model by uncorrelated Gaussian errors with standard deviations $\{\sigma_i\}$, so that the likelihood function for a model predicting $\boldsymbol{\mu}$ is

$$f(\mathbf{y}|\boldsymbol{\mu}) = \prod_i \frac{1}{\sigma_i\sqrt{2\pi}} e^{-(y_i-\mu_i)^2/2\sigma_i^2} = \frac{1}{\sqrt{\det 2\pi\boldsymbol{\sigma}^2}} \exp\left(-\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\sigma}^{-2}(\mathbf{y}-\boldsymbol{\mu})\right) \tag{1.1}$$

The measured values and corresponding uncertainties are in the data file which can be downloaded from `http://ccrg.rit.edu/~whelan/courses/2010_3fa_1060_710/ps07.dat`

a) Consider a model $\mathcal{M}_0$ in which $\mu_i = 0$. Calculate the likelihood $f(\mathbf{y}|\mathcal{M}_0)$ for the data above.

b) Consider an alternative model $\mathcal{M}_1$ in which $\mu_i = \lambda$ where λ is a single parameter obeying a (normalized) uniform prior $f(\lambda|\mathcal{M}_1)$ ranging over values $-5 < \lambda < 5$. Calculate the marginalized likelihood

$$f(\mathbf{y}|\mathcal{M}_1) = \int_{-5}^{5} d\lambda\, f(\mathbf{y}|\lambda,\mathcal{M}_1)\, f(\lambda|\mathcal{M}_1) \tag{1.2}$$

(You may do this by numerical integration, or a semi-analytic calculation involving the error function.)

c) Calculate the Bayes factor $\mathcal{B}_{10} = f(\mathbf{y}|\mathcal{M}_1)/f(\mathbf{y}|\mathcal{M}_0)$ relating the two models.

d) Find (either analytically or numerically) the value $\hat{\lambda}$ which maximizes the likelihood $f(\mathbf{y}|\lambda,\mathcal{M}_1)$ and calculate the ratio

$$\frac{f(\mathbf{y}|\hat{\lambda},\mathcal{M}_1)}{f(\mathbf{y}|\mathcal{M}_0)} \tag{1.3}$$

Compare this to the results of part c) and comment on the effect of the Occam factor in this problem.

e) Extra credit: calculate the χ^2 statistic for each of the two models and comment on the corresponding frequentist comparison between the models.

2 Central Limit Theorem

Consider a uniformly-distributed random variable X, with the pdf

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

a) Calculate

$$\mu_X = \langle X \rangle \tag{2.2a}$$

$$\sigma_X^2 = \left\langle X^2 \right\rangle - \mu_X^2 \tag{2.2b}$$

b) What is the pdf $f_Z(z)$ of the random variable $Z = (X - \mu_X)/\sigma_X$?

c) Consider

$$T = \sum_{k=0}^{N-1} X_k \ , \tag{2.3}$$

the sum of N independent random variables, each distributed according to (2.1). Calculate

$$\mu_T = \langle T \rangle \tag{2.4a}$$

$$\sigma_T^2 = \left\langle T^2 \right\rangle - \mu_T^2 \tag{2.4b}$$

d) Use the Central Limit Theorem to write an approximation for the pdf $f_T(t)$, valid for large N.

e) Extra credit: experimentally check the validity of this approximation for $N = 20$ by randomly generating a large number of t values (each being the sum of twenty uniform random deviates) and plotting their histogram on the same set of axes as the approximate pdf arising from the Central Limit Theorem.

3 Monte Carlo Design

Propose a simple monte carlo experiment to test a statistical method decribed in class or in the textbooks in a quasi-realistic scenario. Describe how you'd simulate the data, what analysis you'd perform, and what properties you'd test. Try not to make any more simplifying assumptions than necessary in the simulation stage; one of the things the monte carlo can test is the validity of any approximations in the analysis stage.