1060-710
Mathematical and Statistical Methods for Astrophysics

Problem Set 5

Assigned 2010 October 12
Due 2010 October 19

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems. Note that if using an outside source to do a calculation, you should use it as a reference for the method, and actually carry out the calculation yourself; it's not sufficient to quote the results of a calculation contained in an outside source.

1 Exercises in Logic and Probability

Do all of the problems at the end of Chapter 2 of Gregory.

2 Change of Variables

Define traditional spherical coördinates (θ, ϕ), with θ being the angle down from the zenith (so that $\theta = 0$ is the zenith and $\theta = \pi/2$ is the horizon) and ϕ being an azimuthal angle which runs from 0 to 2π. Consider an event which occurs at a random sky location above the horizon. The joint pdf for the random variables Θ and Φ is

$$f_{\Theta\Phi}(\theta, \phi) = \frac{\sin\theta}{2\pi} \qquad 0 \le \theta \le \pi/2;\ 0 \le \phi < 2\pi \tag{2.1}$$

a) Integrate over θ and ϕ to confirm that $f_{\Theta\Phi}(\theta, \phi)$ is a normalized density in those variables.

b) Explain why (2.1) represents an isotropic probability distribution.

c) Define new random variables

$$N_x = \sin\Theta\cos\Phi \tag{2.2a}$$
$$N_y = \sin\Theta\sin\Phi \tag{2.2b}$$

be the projections onto two horizontal directions of the unit vector pointing to the event. Perform a change of variables to obtain the joint pdf $f_{N_xN_y}(n_x, n_y)$ for those two variables. (Your answer should not contain θ or ϕ, although they may be convenient for intermediate steps.)

d) What is the region of the (n_x, n_y) plane which corresponds to $0 \le \theta \le \pi/2$, $0 \le \phi < 2\pi$?

e) Marginalize over n_y to find $f_{N_x}(n_x)$.

3 Binomial Distribution

Consider a random event that has a probability of α of occurring in a given trial (e.g., detection of a simulated signal by an analysis pipeline, where α is the efficiency). If we perform n trials, the total number of successes is a binomial random variable K, and its probability mass function is

$$p(k|\alpha, n) = \frac{n!}{k!(n-k)!}\,\alpha^k\,(1-\alpha)^{n-k}; \qquad k = 0, 1, \ldots n \tag{3.1}$$

a) Show that $p(k|\alpha, n)$ is properly normalized, i.e., that

$$\sum_{k=0}^{n} p(k|\alpha, n) = 1 \tag{3.2}$$

(Note that the sum is from 0 to n rather than from 0 to $n-1$, because the number of successes K can be any integer between zero and the total number of trials, inclusive.)

b) Show that the expectation value of the number of successful trials is $n\alpha$.

$$\langle K \rangle = \sum_{k=0}^{n} k\,p(k|\alpha, n) = n\alpha \tag{3.3}$$

(Hint: factor an $n\alpha$ out of the sum and then change variables in what remains to $n' = n-1$ and $k' = k-1$ and show that it sums to unity.)

c) Show that the expected variance in the number of successes is

$$\left\langle K^2 \right\rangle - \langle K \rangle^2 = n\alpha(1-\alpha) \tag{3.4}$$

d) Evaluate the expected mean $\langle K/n \rangle$ and standard deviation $\sqrt{\langle (K/n)^2 \rangle - \langle K/n \rangle^2}$ of the fraction K/n of successful trials. (This is not a trick question; n is not a random variable, so you're really just adjusting the scale to get a fraction.)

e) If we have done the experiment without knowing α, and obtained k successes in n trials, a standard frequentist method is to use the results of part d) to produce an estimate of α with an associated error estimate. What happens to the frequentist estimate and its associated error when $k = 0$? When $k = n$?

f) In a Bayesian method, we use the actual number of successes k in n trials to produce a pdf for the unknown efficiency α. Given a uniform prior $f(\alpha) = 1$, show that the overall probability of k successes in n trials, regardless of the value of α, is

$$p(k|n) := \int_0^1 p(k|\alpha, n)\,f(\alpha)\,d\alpha = \frac{1}{n+1} \tag{3.5}$$

g) Use Bayes's theorem and the result of part f) to show that

$$f(\alpha|k,n) = \frac{(n+1)!}{k!(n-k)!}\,\alpha^k\,(1-\alpha)^{n-k} \qquad 0 < \alpha < 1 \tag{3.6}$$

h) Plot $f(\alpha|0,4)$, $f(\alpha|2,4)$, and $f(\alpha|4,4)$. Comment on the relationship of these posteriors to the corresponding frequentist error estimates when $n = 4$ and $k = 0$, 2, and 4, respectively.

i) Extra credit: find the expectation value $\langle\alpha\rangle = \int_0^1 \alpha\, f(\alpha|k,n)\, d\alpha$ and the standard deviation $\sqrt{\langle\alpha^2\rangle - \langle\alpha\rangle^2}$ associated with the posterior $f(\alpha|k,n)$, expressed in terms of k and n.