1060-710
Mathematical and Statistical Methods for Astrophysics

Problem Set 4

Assigned 2010 September 28
Due 2010 October 5

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems. Note that if using an outside source to do a calculation, you should use it as a reference for the method, and actually carry out the calculation yourself; it's not sufficient to quote the results of a calculation contained in an outside source.

0 Conventions

The convention we're using for the continuous Fourier transform is

$$\mathcal{F}\left\{h\right\} = \widetilde{h}(f) = \int_{-\infty}^{\infty} h(t)\, e^{-i2\pi f t}\, dt \tag{0.1a}$$

$$\mathcal{F}^{-1}\left\{\widetilde{h}\right\} = h(t) = \int_{-\infty}^{\infty} \widetilde{h}(f)\, e^{i2\pi f t}\, df \tag{0.1b}$$

and for the discrete Fourier transform

$$\widehat{h}_k = \sum_{j=0}^{N-1} h_j\, e^{-i2\pi jk/N} \tag{0.2a}$$

$$h_j = \frac{1}{N} \sum_{k=0}^{N-1} \widehat{h}_k\, e^{i2\pi jk/N} \tag{0.2b}$$

1 Colored Noise

This problem should be done in your favorite numerical analysis environment. Please turn in a printout of the final set of commands you used as well as the plots themselves.

Consider a $T = 16\,\text{sec}$ of data sampled at $\frac{1}{\delta t} = 1024\,\text{Hz}$, so that $N = 16\,384$.

a) Generate an N-point series $\{x_j\}$ of random samples drawn from a distribution with zero mean and unit variance, i.e., $\langle x_j \rangle = 0$ and $\langle x_j x_\ell \rangle = \delta_{j\ell}$. (This can be done in matplotlib with `randn(N)`.) Plot x_j versus $t_j - t_0 = j\delta t$ for $t_j - t_0$ from 0 to 16 sec.

b) Take the discrete Fourier transform $\widehat{x}_k$ and plot $|\widehat{x}_k|$ versus $f_k = k\delta f = k/T$ for f_k from $-512\,\text{Hz}$ to $512\,\text{Hz}$.

c) Calculate the average

$$\frac{1}{N} \sum_{k=-N/2}^{N/2-1} |\widehat{x}_k|^2 \tag{1.1}$$

How does it compare to your theoretical expectations for $\left\langle |\widehat{x}_k|^2 \right\rangle$?

d) Construct $\widehat{y}_k = \widehat{x}_k\, e^{-f_k^2/2\sigma_f^2}$, where $\sigma_f = 16\,\text{Hz}$, for $k \in [-N/2, N/2-1]$ and plot $|\widehat{y}_k|$ versus f_k for f_k from $-512\,\text{Hz}$ to $512\,\text{Hz}$.

e) Take the inverse discrete Fourier Transform to get y_j and plot y_j versus $t_j - t_0$ for $t_j - t_0$ from 0 to 16 sec.

2 Power Spectral Density

Consider a single random variable ψ which is equally likely to fall anywhere between 0 and 2π, so that expectation values of random variables whose only randomness comes from ψ can be calculated as

$$\langle F(\psi) \rangle = \frac{1}{2\pi} \int_0^{2\pi} F(\psi)\, d\psi \ . \tag{2.1}$$

Let $x(t) = A \sin(2\pi f_0 t + \psi)$ for some fixed A and f_0.

a) Find $\langle x(t) \rangle$ and $\langle x(t) x(t') \rangle$ and show that $x(t)$ is wide-sense stationary.

b) Find the power spectral density $P_x(f)$.

3 Windowing in PSD Estimates

(Note: In this problem you should repeat analogous calculations from the class notes as necessary, rather than just quoting the results.) The problem of spectral leakage can be reduced by multiplying the time series $\{h_j = h(t_0 + j\,\delta t)\}$ by a window function $\{w_j\}$ before performing the discrete Fourier transform:

$$\widehat{h}^w_k = \sum_{j=0}^{N-1} w_j\, h_j\, e^{-i2\pi jk/N} \ . \tag{3.1}$$

a) Use the inverse Fourier transform

$$h(t) = \int_{-\infty}^{\infty} \widetilde{h}(f)\, e^{i2\pi f(t-t_0)}\, df \tag{3.2}$$

to find an expression for discrete Fourier transform of the windowed data of the form

$$\widehat{h}_k^w = \int_{-\infty}^{\infty} \Lambda([f_k - f]\delta t)\, \widetilde{h}(f)\, df \tag{3.3}$$

where $\Lambda(x)$ is some function constructed using the N-point window $\{w_j\}$, which you will determine.

b) Use (3.3) to find the expectation value $\left\langle \left|\widehat{h}_k^w\right|^2 \right\rangle$ in terms of the PSD $S_h(f)$ and the function $\Lambda(x)$.

c) Use the fact that, for integer j and ℓ,

$$\int_{-1/2}^{1/2} e^{-i2\pi(j-\ell)x}\, dx = \delta_{j\ell} \tag{3.4}$$

to evaluate

$$\int_{-1/2}^{1/2} |\Lambda(x)|^2\, dx \tag{3.5}$$

in terms of the mean square window value

$$\overline{w^2} = \frac{1}{N} \sum_{j=0}^{N-1} (w_j)^2 \tag{3.6}$$

d) Assuming that $|\Lambda(x)|^2$ is sharply enough peaked to be treated as a sum of approximate Dirac delta functions

$$|\Lambda(x)|^2 \approx \mathcal{A} \sum_{s=-\infty}^{\infty} \delta(x+s) \tag{3.7}$$

find the proportionality constant $\mathcal{A}$.

e) Using the approximation (3.7) (with the explicit value you found for $\mathcal{A}$), write an approximate expression for the expectation value $\left\langle \left|\widehat{h}_k^w\right|^2 \right\rangle$ in terms of $S_h(f_k)$, δt, N, and $\overline{w^2}$.

f) Use the result of part f) to write a windowed periodogram P_k^w constructed from $\left|\widehat{h}_k^w\right|^2$ such that $\langle P_k^w \rangle \approx S_h(f_k)$.