1060-710
Mathematical and Statistical Methods for Astrophysics

Problem Set 3

Assigned 2010 September 21
Due 2010 September 28

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems.

1 Chebyshev Polynomials

a) Show that $y = T_n(x)$ is a solution to the Chebyshev differential equation

$$(1 - x^2)y'' - xy' + n^2 y = 0 \tag{1.1}$$

b) Show that the Chebyshev polynomials of the second kind satisfy the following continuous orthogonality relation:

$$\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1 - x^2}\, dx = \begin{cases} 0 & n \neq m \\ \pi/2 & n = m \end{cases} \tag{1.2}$$

c) One can define a *generating function* for a set of orthogonal functions as follows: We remark that

$$\frac{1 - tx}{1 - 2tx + t^2} = \sum_{n=0}^{\infty} T_n(x)t^n = T_0(x) + T_1(x)t + T_2(x)t^2 + T_3(x)t^3 + \dots \tag{1.3}$$

and expand the left hand side as a Taylor series. We then identify terms containing the same power of t on both sides. It might help to remember that $\frac{1}{1-x} = 1 + x + x^2 + \dots$ and thus if $f(x)$ is a polynomial, that $\frac{1}{1-f(x)} = 1 + f(x) + [f(x)]^2 + [f(x)]^3 + \dots$. Here, we have a polynomial in t and x, but the same rules apply. Use this method to demonstrate that the generating function reproduces the proper values of $T_0(x)$, $T_1(x)$, $T_2(x)$, and $T_3(x)$.

d) Show explicitly that if we break down a quadratic $Ax^2 + Bx + C$ defined on the domain $[-1, 1]$ into T_0, T_1, and T_2 modes using the discrete orthogonality relation at the $n = 2$ Gauss-Lobatto points, that the resulting polynomial expansion is exactly equal to the original polynomial, i.e., solve for c_0, c_1, c_2 in terms of A, B, C using the discrete orthogonality relation, and then show that

$$c_0 T_0(x) + c_1 T_1(x) + c_2 T_2(x) = Ax^2 + Bx + C \tag{1.4}$$

Bonus: Can you prove that if $f(x)$ is a polynomial of order k that the expansion into modes based on the $n = k$ Gauss-Lobatto points yields the exact polynomial we started with? Needless to say, there is a logical argument to demonstrate this, not a calculational one.

2 Multipoles

If we assume that the Earth is a uniform-density oblate spheroid with equatorial radius 6378km and polar radius 6356km, then what is its quadrupole moment

$$Q_{zz} = \iiint (3z^2 - r^2)\, \rho\, d^3V \ ? \tag{2.1}$$

What is the approximate magnitude, neglecting the angular terms, of the quadrupole component of the gravitational potential at roughly the moon's distance, 400,000 km, expressed as a fraction of the monopole contribution to the potential?