1016-351-70
Probability

In-class exercise

2010 April 13

Consider a continuous random variable with the uniform probability density function

$$f(x) = \begin{cases} \frac{1}{B-A} & A < x < B \\ 0 & \textbf{otherwise} \end{cases}$$

a. Verify that $f(x)$ is normalized, i.e., that

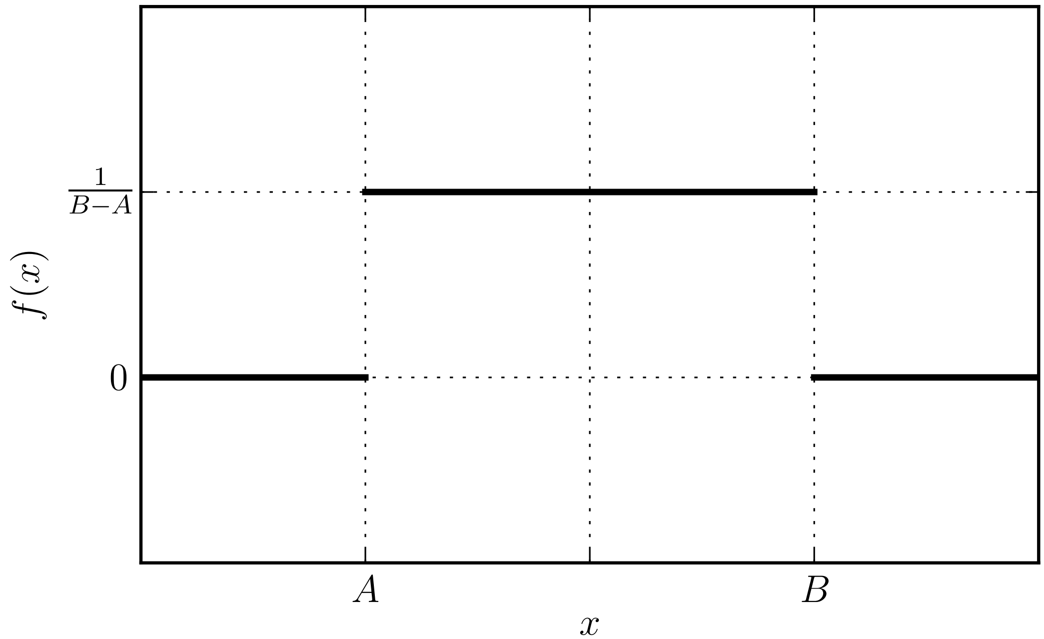
$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

The integral is

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{A} f(x)\,dx + \int_{A}^{B} f(x)\,dx + \int_{B}^{\infty} f(x)\,dx = 0 + \int_{A}^{B} \frac{1}{B-A}\,dx + 0$$
$$= 0 + \left.\frac{x}{B-A}\right|_{A}^{B} + 0 = \frac{B-A}{B-A} = 1$$

so $f(x)$ is indeed normalized.

b. Sketch the graph of $f(x)$. Label the axes.



(There are other possible choices of which tickmarks get which labels, but I wanted the 0 values to be clearly visible.) Note that the pdf is discontinuous, which is fine. Note also that it doesn't really matter what value the pdf $f(x)$ takes on at those points ($x = A$ and $x = B$), since $f(x)$ always gets put under an integral to convert it into a probability.

c. Find the cumulative distribution $F(x)$.

The cdf is the probability

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)\, dy$$

Note that we have to call the integration variable y rather than x because x is the upper limit of the integral. If you have a *definite* integral, the integration variable should never appear in the limits of integration, nor anywhere outside the integral.

Because $f(x)$ has a different form for $x \leq A$, for $A \leq x \leq B$, and for $B \leq x$, the results of the integral will be different depending on where x lies:

$$\text{If } x \leq A, \qquad F(x) = \int_{-\infty}^{x} f(y)\, dy = 0$$

because the integrand is zero over the whole range of integration.

$$\text{If } A \leq x \leq B, \qquad F(x) = \int_{-\infty}^{A} f(y)\, dy + \int_{A}^{x} f(y)\, dy = 0 + \left.\frac{y}{B-A}\right|_{A}^{x} = \frac{x-A}{B-A}$$

Finally,

$$\text{If } B \le x, \qquad F(x) = \int_{-\infty}^{A} f(y)\,dy + \int_{A}^{B} f(y)\,dy + \int_{B}^{x} f(y)\,dy = 0 + 1 + 0 = 1$$

Putting it all together,

$$F(x) = \begin{cases} 0 & x \le A \\ \frac{x-A}{B-A} & A \le x \le B \\ 1 & B \le x \end{cases}$$

Alternate solution using indefinite integrals:

Note that you can also do this by noting that since $F'(x) = f(x)$,

$$F(x) = \int f(x)\,dx$$

where now this is an indefinite integral. Then we have to take the antiderivative of the form of $f(x)$ in each interval:

$$\text{If } x \le A, \qquad F(x) = \int 0\,dx = C_1$$

$$\text{If } A \le x \le B, \qquad F(x) = \int \frac{1}{B-A}\,dx = \frac{x}{B-A} + C_2$$

$$\text{If } B \le x, \qquad F(x) = \int 0\,dx = C_3$$

Because these are indefinite integrals, we have to include an arbitrary constant (C_1, C_2 and C_3, respectively), which is in general different for each integral. Then we need to find the values of these constants which ensure that $F(-\infty) = 0$ and that $F(x)$ is continuous, which it must be for a continuous random variable. (We can then check that $F(\infty) = 1$, which must be the case if the pdf $f(x)$ was properly normalized, and we didn't make any mistakes.) We find C_1 from

$$F(-\infty) = C_1 = 0$$

so that

$$\text{If } x \le A, \qquad F(x) = 0$$

and then find C_2 from continuity at $x = A$

$$F(A) = 0 = \frac{A}{B-A} + C_2$$

so that

$$C_2 = -\frac{A}{B-A}$$

and

$$\text{if } A \le x \le B, \qquad F(x) = \frac{x}{B-A} - \frac{A}{B-A}$$

We then find C_3 from continuity at $x = B$

$$F(B) = \frac{B}{B-A} - \frac{A}{B-A} = 1 = C_3$$

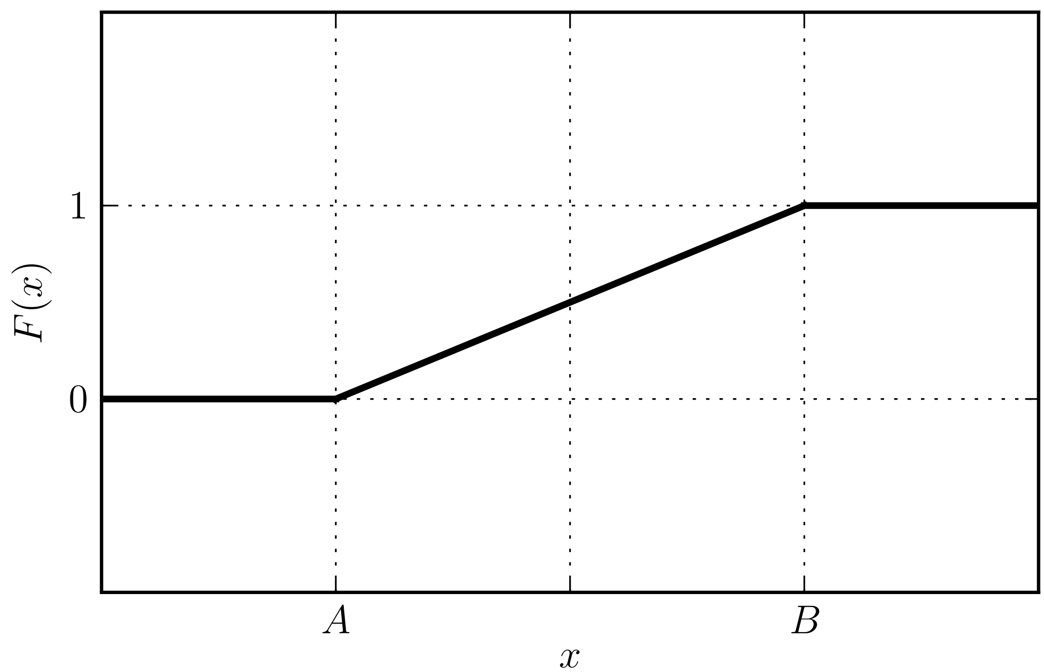
and thus

$$\text{If } B \leq x, \qquad F(x) = 1$$

Finally, we can then see that $F(\infty) = 1$, so everything is consistent.

In the end the indefinite integral approach gives the right answer (of course) if you're careful about the integration constants. Personally, I find the approach with definite integrals to be easier, since the matching happens automatically.

d. Sketch the graph of $F(x)$. Label the axes.



Notice that the graph is continuous, as it must be for a continuous random variable.

e. Calculate the expected value $E(X)$ in terms of A and B.

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_A^B \frac{x}{B-A}\, dx = \frac{1}{B-A}\, \frac{x^2}{2}\bigg|_A^B = \frac{B^2 - A^2}{2(B-A)}$$
$$= \frac{(B+A)(B-A)}{2(B-A)} = \frac{A+B}{2}$$

f. Calculate the variance $V(X)$ in terms of A and B.

The easiest way to do this is to calculate

$$
\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} x^2\, f(x)\, dx = \int_A^B \frac{x^2}{B-A}\, dx = \frac{1}{B-A}\, \frac{x^3}{3}\bigg|_A^B = \frac{B^3 - A^3}{3(B-A)} \\
&= \frac{(B^2 + AB + A^2)(B-A)}{3(B-A)} = \frac{B^2 + AB + A^2}{3}
\end{aligned}
$$

And then

$$
\begin{aligned}
V(X) = E(X^2) - \big(E(X)\big)^2 &= \frac{B^2 + AB + A^2}{3} - \left(\frac{A+B}{2}\right)^2 = \frac{B^2 + AB + A^2}{3} - \frac{A^2 + 2AB + B^2}{4} \\
&= \frac{4B^2 + 4AB + 4A^2 - 3A^2 - 6AB - 3B^2}{12} = \frac{B^2 - 2AB + A^2}{12} = \frac{(B-A)^2}{12}
\end{aligned}
$$