Lagrangian and Hamiltonian Mechanics (Symon Chapter Nine)

Physics A301*

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Contents

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1 Lagrangian Mechanics

Newtonian mechanics is most naturally written in terms of inertial Cartesian coördinates of one or more particles. But often it's more convenient to describe a problem in terms of a different set of variables, such as spherical or other non-Cartesian coördinates, non-inertial coördinates, or even variables which combine the coördinates of different particles (e.g. the center-of-mass and relative position vectors in the two-body problem). In each of those cases, we've had to start with the inertial Cartesian formulation, and deduce the form of the equations of motion in terms of the more general sets of coördinates. The fundamental reason why things are so much easier in Cartesian coördinates is the vector nature of the equations of motion written in terms of the force $\vec{F}$. It turns out that by concentrating on scalar expressions like the energy, we can formulate mechanics in a way that is equivalent for any set of coördinates, Cartesian or not. That formulation is called *Lagrangian Mechanics*.

1.1 Derivation of the Lagrange Equations

1.1.1 Newton's Second Law from Scalar Functions

First, let's make an observation about Newton's second law:

$$m\ddot{\vec{r}} = \frac{d\vec{p}}{dt} = \vec{F} \tag{1.1}$$

If the force is dependent only on position, and conservative, it can be written in terms of a potential $V(x, y, z)$

$$\vec{F} = -\vec{\nabla} V \tag{1.2}$$

written out in terms of components, this means the components of the force can be written in terms of derivatives of the kinetic energy.

$$F_x = -\frac{\partial V}{\partial x} \tag{1.3a}$$

$$F_y = -\frac{\partial V}{\partial y} \tag{1.3b}$$

$$F_z = -\frac{\partial V}{\partial z} \tag{1.3c}$$

On the other hand, since the kinetic energy is

$$T(\dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}m\left|\dot{\vec{r}}\right|^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2 \tag{1.4}$$

the components of the momentum can be written as partial derivatives of the kinetic energy:

$$p_x = m\dot{x} = \frac{\partial T}{\partial \dot{x}} \tag{1.5a}$$
$$p_y = m\dot{y} = \frac{\partial T}{\partial \dot{y}} \tag{1.5b}$$
$$p_z = m\dot{z} = \frac{\partial T}{\partial \dot{z}} \tag{1.5c}$$

So we can write the equations of motion starting from the scalar functions $T(\dot{x}, \dot{y}, \dot{z})$ and $V(x, y, z)$:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = -\frac{\partial V}{\partial x} \tag{1.6a}$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) = -\frac{\partial V}{\partial y} \tag{1.6b}$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) = -\frac{\partial V}{\partial z} \tag{1.6c}$$

For a system of N particles with forces determined by an overall potential energy $V(\{x_k\}, \{y_k\}, \{z_k\})$, everything goes as before and the equations of motion can be written in terms of derivatives of V and the total kinetic energy $T(\{\dot{x}_k\}, \{\dot{y}_k\}, \{\dot{z}_k\})$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_k}\right) = -\frac{\partial V}{\partial x_k} \tag{1.7a}$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_k}\right) = -\frac{\partial V}{\partial y_k} \tag{1.7b}$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}_k}\right) = -\frac{\partial V}{\partial z_k} \tag{1.7c}$$

1.1.2 Non-Cartesian Examples

So far, this seems like nothing but an intellectual curiosity, but let's see what happens if we try to write Newton's third law from scratch in non-Cartesian or non-inertial coördinates.

Polar Coördinates Consider a two-dimensional problem where we define polar coördinates r and ϕ according to the usual convention

$$x = r\cos\phi \tag{1.8a}$$
$$y = r\sin\phi \tag{1.8b}$$

We can't write down the equations of motion directly in these coördinates:

$$m\ddot{r} \neq \hat{r} \cdot \vec{F} \tag{1.9a}$$
$$m\ddot{\phi} \neq \hat{\phi} \cdot \vec{F} \tag{1.9b}$$

because Newton's second law takes the form

$$\vec{F} = m\ddot{\vec{r}} = m(\ddot{r} - r\dot{\phi}^2)\hat{r} + m(r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi} = F_r\hat{r} + F_\phi\hat{\phi} \tag{1.10}$$

with the extra terms arising because of the non-constant polar coördinate basis vectors:

$$\frac{d\hat{r}}{dt} \neq \vec{0} \tag{1.11a}$$

$$\frac{d\hat{\phi}}{dt} \neq \vec{0} \tag{1.11b}$$

The equations of motion are:

$$m\ddot{r} = F_r + mr\dot{\phi}^2 \tag{1.12a}$$

$$mr\ddot{\phi} = F_\phi - 2m\dot{r}\dot{\phi} \tag{1.12b}$$

Rotating Coördinates Consider another two-dimensional problem where we find it convenient to define coördinates rotating with angular speed ω relative to the inertial Cartesian coördinates. Those might be written explicitly as

$$x^* = x\cos\omega t + y\sin\omega t \tag{1.13a}$$

$$y^* = -x\sin\omega t + y\cos\omega t \tag{1.13b}$$

again, the form of Newton's second law is not the simplest one:

$$m\ddot{x}^* \neq \hat{x}^* \cdot \vec{F} \tag{1.14a}$$

$$m\ddot{y}^* \neq \hat{y}^* \cdot \vec{F} \tag{1.14b}$$

because of the rotation of the starred basis vectors. Instead, the equations of motion come from

$$m\ddot{x}^*\hat{x}^* + m\ddot{y}^*\hat{y}^* = m\frac{d^{*2}\vec{r}}{dt} = \vec{F} - \vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2\vec{\omega} \times \frac{d^*\vec{r}}{dt} \tag{1.15}$$

where $\vec{\omega} = \omega\hat{z}$

So again, the equations of motion contain correction terms compared to the Cartesian forms.

Two-Body Problem In this case, the coördinates of the particles are mixed up in the components of the vectors

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \tag{1.16a}$$

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \tag{1.16b}$$

Here the situation is even more extreme than just having different basis vectors. It's not even clear-cut how you define the "component" of force corresponding to, for example, x or Z.

1.1.3 Generalized Coördinates

In all those cases, we would get the equations of motion by starting from the Cartesian form of Newton's second law and converting the resulting equations of motion. We'd like, instead, to find a generic method, starting with kinetic and potential energy, to construct the equations of motion directly in *any* set of coördinates.

So, suppose we have N particles moving in three dimensions. There are then a total of $3N$ Cartesian coördinates: x_1, y_1, z_1, $x_2 \ldots y_N$, and z_N. For notational convenience, we'll define the symbol X_ℓ to refer to any of the $3N$ Cartesian coördinates, labelled as follows:

$$X_1 = x_1, \quad X_2 = y_1, \quad X_3 = z_1, \quad X_4 = x_2, \quad \ldots, \quad X_{3N} = z_N \tag{1.17}$$

We refer to the whole list of $3N$ Cartesian coördinates by any of the following:

$$\{x_1, y_1, \ldots, z_N\} \equiv \{X_1, X_2, \ldots, X_{3N}\} \equiv \{X_\ell | \ell = 1 \ldots 3N\} \equiv \{X_\ell\} \tag{1.18}$$

We can also describe the same system a different set of $3N$ *generalized coördinates*, which we refer to as

$$\{q_1, q_2, \ldots, q_{3N}\} \equiv \{q_k | k = 1 \ldots 3N\} \equiv \{q_k\} \tag{1.19}$$

If this is a good set of coördinates, we should be able to work out the generalized coördinates of a particle given its Cartesian coördinates, although that relationship may be time-dependent. So we can write each of the generalized coördinates as a function of the full set of Cartesian coördinates and time:

$$q_k = q_k(\{X_\ell\}, t) \tag{1.20}$$

and if this is a good coördinate mapping, we should also be able to invert the relationship and write each Cartesian coördinate as a function of the full set of generalized coördinates and time:

$$X_\ell = X_\ell(\{q_k\}, t) \tag{1.21}$$

Example: Polar Coördinates In this case, it's only a two-dimensional problem, so "$3N$", the number of Cartesian coördinates is only 2: $X_1 = x$ and $X_2 = y$. The generalized coördinates are[1]

$$q_1 = r = \sqrt{x^2 + y^2} \tag{1.22a}$$

$$q_2 = \phi = \tan^{-1}\left(\frac{y}{x}\right) \tag{1.22b}$$

The inverse transformation is

$$X_1 = x = r\cos\phi \tag{1.23a}$$

$$X_2 = y = r\sin\phi \tag{1.23b}$$

[1] Technically, the expression for ϕ is a little more complicated, since the principal value of the arctangent is between $-\pi/2$ and $\pi/2$, which is only the case if $x \geq 0$; we should actually specify something which adds or subtracts π if $x < 0$ to give the correct angle for negative x. Computer programming languages define such a function, and the correct expression is `phi=atan2(y,x)`.

Example: Rotating Coördinates Once again, in this two-dimensional problem, "$3N$"=2. Now, the coördinate transformation is explicitly time-dependent:

$$q_1 = x^*(x, y, t) = x \cos \omega t + y \sin \omega t \tag{1.24a}$$
$$q_2 = y^*(x, y, t) = -x \sin \omega t + y \cos \omega t \tag{1.24b}$$

1.1.4 Coördinate Transformations and the Chain Rule

Now we want look for a set of equations of motion in generalized coördinates, something like the equations of motion (1.7) in Cartesian coördinates, which in our streamlined notation are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{X}_\ell}\right) = -\frac{\partial V}{\partial X_\ell} \tag{1.25}$$

note that this holds for any value of ℓ, so it represents all $3N$ equations of motion.

Our strategy is going to be to work out the corresponding partial derivatives of the scalar functions

$$V(\{q_k\}, t) = V(\{X_\ell(\{q_k\}, t)\}, t) \tag{1.26}$$

and

$$T(\{q_k\}, \{\dot{q}_k\}, t) = T(\{\dot{X}_\ell(\{q_k\}, \{\dot{q}_k\}, t)\}) \tag{1.27}$$

in term of the derivatives $\left\{\frac{\partial V}{\partial X_\ell}\right\}$ and $\left\{\frac{\partial T}{\partial \dot{X}_\ell}\right\}$ and see what the consequences of (1.27) are.

Derivatives of Potential Energy Let's start with the potential energy, since it's simpler. We want the partial derivative $\frac{\partial V}{\partial q_k}$, which is the derivative with respect to one particular q_k (for one value of k) with all the other $q_{k'}$s (for all other values of the index), as well as time, treated as constants. The chain rule tells us how to calculate this derivative, if we realize that V's dependence on q_k comes through the q_k dependence of all the $\{X_\ell\}$s:

$$\frac{\partial V}{\partial q_k} = \frac{\partial V}{\partial X_1}\frac{\partial X_1}{\partial q_k} + \frac{\partial V}{\partial X_2}\frac{\partial X_2}{\partial q_k} + \ldots + \frac{\partial V}{\partial X_{3N}}\frac{\partial X_{3N}}{\partial q_k} = \sum_{\ell=1}^{3N}\frac{\partial V}{\partial X_\ell}\frac{\partial X_\ell}{\partial q_k} \tag{1.28}$$

Total and Partial Time Derivatives So far, so good. Things get a little trickier with the kinetic energy, since we have to consider the velocity $\dot{X}_\ell$. In a sense we have two different perspectives on the behavior of X_ℓ as a function. On the one hand, the inverse coördinate transformations define it as just a function of some arguments q_1, q_2, $\ldots q_{3N}$, and t. We can take a partial derivative with respect to any of those arguments, treating all the others, formally, as constants. That is the meaning of the partial derivative

$$\frac{\partial X_\ell}{\partial t} = \left(\frac{\partial X_\ell}{\partial t}\right)_{\{q_k\}} \tag{1.29}$$

On the other hand, X_ℓ, as a physical quantity, has some actual time dependence determined by the trajectories of the particles in the problem. So from that point of view, it's just a function of one variable, time $X_\ell(t)$. The relationship between the two comes about because the actual trajectories of the particles also imply some physical time dependence for the generalized coördinates as well. The time dependence of the Cartesian coördinates can be determined from the time dependence of the generalized coördinates and the coördinate transformation:

$$X_\ell(t) = X_\ell(\{q_k(t)\}, t) \tag{1.30}$$

The velocity $\dot{X}_\ell$ is the time derivative of the actual physical trajectory $X_\ell(t)$ and is this a *total* (not partial) derivative, which can be calculated using the chain rule. Explicitly, if we think about advancing an infinitesimal amount of time dt, all of the generalized coördinates will change by infinitesimal amounts $\{dq_k\}$, inducing a change

$$dX_\ell = \sum_{k=1}^{3N} \frac{\partial X_\ell}{\partial q_k}\, dq_k + \frac{\partial X_\ell}{\partial t}\, dt \tag{1.31}$$

in the Cartesian coördinate. The time derivative is just this infinitesimal change divided by the infinitesimal time change dt:

$$\dot{X}_\ell = \frac{dX_\ell}{dt} = \sum_{k=1}^{3N} \frac{\partial X_\ell}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial X_\ell}{\partial t}\frac{dt}{dt} = \sum_{k=1}^{3N} \frac{\partial X_\ell}{\partial q_k}\dot{q}_k + \frac{\partial X_\ell}{\partial t} \tag{1.32}$$

The partial derivatives $\{\frac{\partial X_\ell}{\partial q_k}\}$ and $\frac{\partial X_\ell}{\partial t}$ are, like X_ℓ, functions of $\{q_k(t)\}$ and t. This means the functional dependence of the Cartesian velocity is $\dot{X}_\ell(\{q_k\}, \{\dot{q}_k\}, t)$.

As an example of how this works, we return to one of our three canonical examples, this time the rotating coördinates $q_1 = x^*$ and $q_2 = y^*$. The inverse coördinate transformations are

$$X_1 = x = x^* \cos\omega t - y^* \sin\omega t \tag{1.33a}$$

$$X_2 = y = x^* \sin\omega t + y^* \cos\omega t \tag{1.33b}$$

Focussing on x, the partial derivatives are

$$\frac{\partial x}{\partial x^*} = \left(\frac{\partial x}{\partial x^*}\right)_{y^*,t} = \cos\omega t \tag{1.34a}$$

$$\frac{\partial x}{\partial y^*} = \left(\frac{\partial x}{\partial y^*}\right)_{x^*,t} = -\sin\omega t \tag{1.34b}$$

$$\frac{\partial x}{\partial t} = \left(\frac{\partial x}{\partial t}\right)_{x^*,y^*} = -\omega x^* \sin\omega t - \omega y^* \cos\omega t \tag{1.34c}$$

On the other hand, the x component of the velocity, which is the total time derivative, is

$$\dot{x} = \underbrace{\dot{x}^* \cos\omega t}_{\frac{\partial x}{\partial x^*}\frac{dx^*}{dt}} \underbrace{- \dot{y}^* \sin\omega t}_{\frac{\partial x}{\partial y^*}\frac{dy^*}{dt}} \underbrace{- \omega x^* \sin\omega t - \omega y^* \cos\omega t}_{\frac{\partial x}{\partial t}} \tag{1.35}$$

The corresponding derivatives of y are left as an exercise.

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1.1.5 Partial Derivatives of the Kinetic and Potential Energy

In Cartesian coördinates, the kinetic energy T is a function only of the velocities $\{\dot{X}_\ell\}$:

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_1\dot{z}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \ldots \frac{1}{2}m_N\dot{z}_N^2 = \sum_{\ell=1}^{3N} \frac{1}{2}M_\ell \dot{X}_\ell^2 \tag{1.36}$$

(where we have defined the notation $M_1 = M_2 = M_3 = m_1$, $M_4 = M_5 = M_6 = m_2$, $\ldots M_{3N-2} = M_{3N-1} = M_{3N} = m_N$). But since the velocities $\{\dot{X}_\ell\}$ are functions in general of the generalized coördinates $\{q_k\}$ and time t as well as the generalized velocities $\{\dot{q}_k\}$, the kinetic energy depends on all of those things when written in generalized coördinates:

$$T(\{\dot{X}_\ell\}) = T(\{q_k\}, \{\dot{q}_k\}, t) \tag{1.37}$$

So the derivative we're interested in is $\frac{\partial T}{\partial \dot{q}_k}$, a partial derivative with respect to one $\dot{q}_k$ with all the other $\dot{q}_{k'}$s, plus *all* of the $q_{k'}$s and t treated as constants. The chain rule tells us that

$$\frac{\partial T}{\partial \dot{q}_k} = \sum_{\ell=1}^{3N} \frac{\partial T}{\partial \dot{X}_\ell}\frac{\partial \dot{X}_\ell}{\partial \dot{q}_k} \tag{1.38}$$

So the thing we need to relate one set of partial derivatives to the other is $\frac{\partial \dot{X}_\ell}{\partial \dot{q}_k}$ for each possible choice of ℓ and k. We can actually look explicitly at the $\dot{q}_k$ dependence by writing out (1.32):

$$\dot{X}_\ell = \frac{\partial X_\ell}{\partial q_1}\dot{q}_1 + \frac{\partial X_\ell}{\partial q_2}\dot{q}_2 + \ldots + \frac{\partial X_\ell}{\partial q_{3N}}\dot{q}_{3N} + \frac{\partial X_\ell}{\partial t} \tag{1.39}$$

As we've noted before, all the coëfficients like $\frac{\partial X_\ell}{\partial q_1}$ and $\frac{\partial X_\ell}{\partial t}$ will depend only on $\{q_{k'}\}$ and t, and not on $\dot{q}_k$. In fact, for each possible value of k, only one term contains $\dot{q}_k$; for example, if $k = 5$ the only term which gives a non-zero contribution to the partial derivative $\frac{\partial \dot{X}_\ell}{\partial \dot{q}_5}$ is

$$\frac{\partial \dot{X}_\ell}{\partial \dot{q}_5} = \frac{\partial}{\partial \dot{q}_5}\left(\frac{\partial X_\ell}{\partial q_5}\dot{q}_5\right) = \frac{\partial X_\ell}{\partial q_5} \tag{1.40}$$

So for a general k, we find that

$$\frac{\partial \dot{X}_\ell}{\partial \dot{q}_k} = \frac{\partial X_\ell}{\partial q_k} \tag{1.41}$$

which means that

$$\frac{\partial T}{\partial \dot{q}_k} = \sum_{\ell=1}^{3N} \frac{\partial T}{\partial \dot{X}_\ell}\frac{\partial \dot{X}_\ell}{\partial \dot{q}_k} = \sum_{\ell=1}^{3N} \frac{\partial T}{\partial \dot{X}_\ell}\frac{\partial X_\ell}{\partial q_k} \tag{1.42}$$

So now we have $\frac{\partial T}{\partial \dot{q}_k}$ from (1.42) and $\frac{\partial V}{\partial q_k}$ from (1.28). To write something analogous to (1.27), we need to take the (total) time derivative of this. So we differentiate (1.42), using the sum and product rules:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) &= \sum_{\ell=1}^{3N}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{X}_\ell}\right)\frac{\partial X_\ell}{\partial q_k} + \frac{\partial T}{\partial \dot{X}_\ell}\frac{d}{dt}\left(\frac{\partial X_\ell}{\partial q_k}\right)\right] \\ &= \sum_{\ell=1}^{3N}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{X}_\ell}\right)\frac{\partial X_\ell}{\partial q_k} + \sum_{\ell=1}^{3N}\frac{\partial T}{\partial \dot{X}_\ell}\frac{d}{dt}\left(\frac{\partial X_\ell}{\partial q_k}\right)\end{aligned} \tag{1.43}$$

The first sum can be simplified by using the equation of motion (1.27) and the chain rule:

$$\sum_{\ell=1}^{3N}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{X}_\ell}\right)\frac{\partial X_\ell}{\partial q_k} = \sum_{\ell=1}^{3N}\left(-\frac{\partial V}{\partial X_\ell}\right)\frac{\partial X_\ell}{\partial q_k} = -\frac{\partial V}{\partial q_k} \tag{1.44}$$

The second sum, which is the correction arising from the use of generalized coördinates, can be evaluated by noting that $\frac{\partial X_\ell}{\partial q_k}$, like X_ℓ, is a function of all the $q_{k'}$s as well as time, so we can write its total time derivative as

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial X_\ell}{\partial q_k}\right) &= \sum_{k'=1}^{3N}\frac{\partial}{\partial q_{k'}}\left(\frac{\partial X_\ell}{\partial q_k}\right)\frac{dq_{k'}}{dt} + \frac{\partial}{\partial t}\left(\frac{\partial X_\ell}{\partial q_k}\right) = \sum_{k'=1}^{3N}\left(\frac{\partial^2 X_\ell}{\partial q_{k'}\partial q_k}\right)\dot{q}_{k'} + \left(\frac{\partial^2 X_\ell}{\partial t\partial q_k}\right) \\ &= \frac{\partial}{\partial q_k}\underbrace{\left[\sum_{k'=1}^{3N}\left(\frac{\partial X_\ell}{\partial q_{k'}}\right)\dot{q}_{k'} + \left(\frac{\partial X_\ell}{\partial t}\right)\right]}_{\frac{dX_\ell}{dt}} = \frac{\partial \dot{X}_\ell}{\partial q_k}\end{aligned} \tag{1.45}$$

Plugging this back into (1.43) gives us

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = -\frac{\partial V}{\partial q_k} + \sum_{\ell=1}^{3N}\frac{\partial T}{\partial \dot{X}_\ell}\frac{\partial \dot{X}_\ell}{\partial q_k} = -\frac{\partial V}{\partial q_k} + \frac{\partial T}{\partial q_k} \tag{1.46}$$

where in the last step we have used the chain rule applied to

$$T(\{q_k\},\{\dot{q}_k\},t) = T\left(X_\ell(\{\{q_k\},\{\dot{q}_k\},t)\right) \tag{1.47}$$

So this means the equations of motion (Newton's second law) are equivalent to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)\frac{\partial}{\partial q_k}(T-V) \tag{1.48}$$

Now, since V is just a potential energy and depends only on the coördinates and time,

$$\frac{\partial V}{\partial \dot{q}_k} = 0 \tag{1.49}$$

so we can make the equation look more symmetric by adding zero to the left-hand side and writing

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_k}(T-V)\right) = \frac{\partial}{\partial q_k}(T-V) \tag{1.50}$$

The quantity appearing in parentheses on both sides of the equation is called the Lagrangian

$$L = T - V \tag{1.51}$$

Note that this is kinetic *minus* potential energy, as opposed to the total energy, which is kinetic *plus* potential. We've now shown that for any set of generalized coördinates[2], Newton's second law is equivalent to the full set of *Lagrange Equations*:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad k = 1, 2, \ldots, 3N \tag{1.52}$$

This is the generic form we were looking for. All we need to do is write the kinetic and potential energies in terms of the generalized coördinates, their time derivatives, and time, and we can find the equations of motion by taking partial derivatives without ever needing to use inertial Cartesian coördinates.

1.2 Examples

Let's return to our three examples of generalized coördinates and see how we obtain the correct equations of motion in each case:

1.2.1 Polar Coördinates

Recall that here the total number of coördinates "$3N$" is 2, and the Cartesian and generalized coördinates are

$$\begin{aligned} X_1 &= x \qquad & q_1 &= r \\ X_2 &= y \qquad & q_2 &= \phi \end{aligned}$$

The kinetic energy is

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 \tag{1.54}$$

which we can obtain either directly by considering the infinitesimal distance

$$ds^2 = dr^2 + r^2\,d\phi^2 \tag{1.55}$$

associated with changes in r and ϕ, or by starting with

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 \tag{1.56}$$

and making the substitutions

$$\dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi \tag{1.57a}$$

$$\dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi \tag{1.57b}$$

[2]which can in principle be related by a coördinate transformation to the Cartesian coördinates for a problem with forces arising solely from a potential energy

Meanwhile the potential energy $V(r, \phi)$ is a scalar field which can be described equally well as a function of the polar coördinates. So the Lagrangian is

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - V(r,\phi) \tag{1.58}$$

Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \tag{1.59a}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \tag{1.59b}$$

so to apply them we need to take the relevant partial derivatives:

$$\frac{\partial L}{\partial r} = mr\dot{\phi}^2 - \frac{\partial V}{\partial r} \tag{1.60a}$$

$$\frac{\partial L}{\partial \phi} = -\frac{\partial V}{\partial \phi} \tag{1.60b}$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \text{radial compt of momentum} \tag{1.60c}$$

$$\frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} \qquad \text{angular momentum} \tag{1.60d}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r} \tag{1.60e}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = mr^2\ddot{\phi} + 2mr\dot{r}\dot{\phi} \tag{1.60f}$$

So the equations of motion are

$$m\ddot{r} = mr\dot{\phi}^2 - \frac{\partial V}{\partial r} \tag{1.61a}$$

$$mr^2\ddot{\phi} + 2mr\dot{r}\dot{\phi} = -\frac{\partial V}{\partial \phi} \tag{1.61b}$$

We can verify that these are the same equations of motion given by

$$m\ddot{\vec{r}} = -\vec{\nabla}V \tag{1.62}$$

in the vector approach, in which

$$-\vec{\nabla}V = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial \phi}\hat{\phi} \tag{1.63}$$

and, thanks to the position and therefore time-dependent basis vectors

$$\frac{d\hat{r}}{dt} = \dot{\phi}\hat{\phi} \tag{1.64a}$$

$$\frac{d\hat{\phi}}{dt} = -\dot{\phi}\hat{r} \tag{1.64b}$$

the acceleration is calculated via

$$\vec{r} = r\hat{r} \qquad \Rightarrow \qquad \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} \qquad \Rightarrow \qquad \ddot{\vec{r}} = \ddot{r}\hat{r} + 2\dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} - r\dot{\phi}^2\hat{r} \tag{1.65}$$

to give the equations of motion

$$m\ddot{\vec{r}} = \left(m\ddot{r} - mr\dot{\phi}^2\right)\hat{r} + \left(mr\ddot{\phi} + 2m\dot{r}\dot{\phi}\right) \tag{1.66}$$

Equating the r and ϕ components of the vector expressions (1.66) and (1.63) gives the same equations of motion as (1.61).

1.2.2 Rotating Coördinates

Here again the total number of coördinates "$3N$" is 2, and the Cartesian and generalized coördinates are

$$\begin{aligned} X_1 &= x \qquad & q_1 &= x^* \\ X_2 &= y \qquad & q_2 &= y^* \end{aligned}$$

To get the kinetic energy, we really need to start with the inertial form and transform it using

$$\dot{x} = (\dot{x}^* - \omega y^*)\cos\omega t - (\dot{y}^* + \omega x^*)\sin\omega t \tag{1.68a}$$

$$\dot{y} = (\dot{x}^* - \omega y^*)\sin\omega t - (\dot{y}^* + \omega x^*)\cos\omega t \tag{1.68b}$$

which makes the kinetic energy

$$\begin{aligned} T &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2) = \frac{1}{2}m(\dot{x}^* - \omega y^*)^2 + \frac{1}{2}m(\dot{y}^* + \omega x^*)^2 \\ &= \frac{m}{2}\left[(\dot{x}^*)^2 + (\dot{y}^*)^2 - 2\omega\dot{x}^*y^* + 2\omega\dot{y}^*x^* + \omega^2(x^*)^2 + \omega^2(y^*)^2\right] \end{aligned} \tag{1.69}$$

and the Lagrangian

$$L = T - V(x^*, y^*, t) \tag{1.70}$$

so that the partial derivatives are

$$\frac{\partial L}{\partial x^*} = m\omega\dot{y}^* + m\omega^2 x^* - \frac{\partial V}{\partial x^*} \tag{1.71a}$$

$$\frac{\partial L}{\partial y^*} = -m\omega\dot{x}^* + m\omega^2 y^* - \frac{\partial V}{\partial y^*} \tag{1.71b}$$

$$\frac{\partial L}{\partial \dot{x}^*} = m\dot{x}^* - m\omega y^* \tag{1.71c}$$

$$\frac{\partial L}{\partial \dot{y}^*} = m\dot{y}^* + m\omega x^* \tag{1.71d}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^*}\right) = m\ddot{x}^* - m\omega\dot{y}^* \tag{1.71e}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^*}\right) = m\ddot{y}^* + m\omega\dot{y}^* \tag{1.71f}$$

So that the equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^*} - \frac{\partial L}{\partial x^*} = m\ddot{x}^* - \left(2m\omega\dot{y}^* + m\omega^2 x^* - \frac{\partial V}{\partial x^*}\right) = 0 \tag{1.72a}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}^*} - \frac{\partial L}{\partial y^*} = m\ddot{y}^* - \left(-2m\omega\dot{x}^* + m\omega^2 y^* - \frac{\partial V}{\partial y^*}\right) = 0 \tag{1.72b}$$

And one can easily check (exercise!) that these equations of motion are the same as those given by the non-inertial coördinate system method including fictitious forces.

1.2.3 Two-Body Problem

In this case, $3N = 6$ and the Cartesian coördinates are the six components of the two position vectors $\vec{r}_1$ and $\vec{r}_2$, while the generalized coördinates are the six components of the vectors $\vec{R}$ and $\vec{r}$. The most interesting part about the construction of the Lagrangian is the kinetic energy, which can be obtained by differentiating the inverse coördinate transformations:

$$\vec{r}_1 = \vec{R} + \frac{m_2}{m_1+m_2}\vec{r} \tag{1.73a}$$

$$\vec{r}_2 = \vec{R} - \frac{m_1}{m_1+m_2}\vec{r} \tag{1.73b}$$

to get

$$\dot{\vec{r}}_1 = \dot{\vec{R}} + \frac{m_2}{m_1+m_2}\dot{\vec{r}} \tag{1.74a}$$

$$\dot{\vec{r}}_2 = \dot{\vec{R}} - \frac{m_1}{m_1+m_2}\dot{\vec{r}} \tag{1.74b}$$

The kinetic energy is thus

$$\begin{aligned} T = \frac{1}{2}m_1\dot{\vec{r}}_1\cdot\dot{\vec{r}}_1 + \frac{1}{2}m_2\dot{\vec{r}}_2\cdot\dot{\vec{r}}_2 &= \frac{1}{2}m_1\dot{\vec{R}}\cdot\dot{\vec{R}} + \cancel{\frac{m_1m_2}{m_1+m_2}\dot{\vec{r}}\cdot\dot{\vec{R}}} + \frac{1}{2}\frac{m_1m_2^2}{(m_1+m_2)^2}\dot{\vec{r}}\cdot\dot{\vec{r}} \\ &\quad + \frac{1}{2}m_2\dot{\vec{R}}\cdot\dot{\vec{R}} - \cancel{\frac{m_1m_2}{m_1+m_2}\dot{\vec{r}}\cdot\dot{\vec{R}}} + \frac{1}{2}\frac{m_1^2m_2}{(m_1+m_2)^2}\dot{\vec{r}}\cdot\dot{\vec{r}} \\ &= \frac{1}{2}(m_1+m_2)\dot{\vec{R}}\cdot\dot{\vec{R}} + \frac{1}{2}\frac{\cancel{(m_1+m_2)}m_1m_2}{(m_1+m_2)^{\cancel{2}}}\dot{\vec{r}}\cdot\dot{\vec{r}} = \frac{1}{2}M\dot{\vec{R}}\cdot\dot{\vec{R}} + \frac{1}{2}\mu\dot{\vec{r}}\cdot\dot{\vec{r}} \end{aligned} \tag{1.75}$$

And this is actually a more direct way of showing the equivalence to the one-body problem. Furthermore, if the potential energy depends only on the distance $|\vec{r}_1 - \vec{r}_2| = |\vec{r}| = r$ between the two bodies (central force motion with no external forces), the Lagrangian becomes

$$L = \frac{1}{2}M\dot{\vec{R}}\cdot\dot{\vec{R}} + \frac{1}{2}\mu\dot{\vec{r}}\cdot\dot{\vec{r}} - V(r) \tag{1.76}$$

Supplemental Notes

The following notes follow the presentation used in previous years, which may or may not follow Dr. Brans's treatment. They are included as a potentially useful resource.

2 Lagrangian Formulation with Constraints

2.0 Recap

Recall that a system of N particles moving in 3 dimensions has a set of $3N$ Cartesian coördinates $\{X_\ell\}$:

$$X_1 = x_1, \qquad X_2 = y_1, \qquad X_3 = z_1, \qquad \ldots \qquad X_{3N} = z_N \tag{2.1}$$

Newton's second law in in the presence of a potential V is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{X}_\ell}\right) = M_\ell \ddot{X}_\ell = -\frac{\partial V}{\partial X_\ell} \qquad \ell = 1, 2, \ldots 3N \tag{2.2}$$

where

$$T(\{\dot{X}_\ell\}) = \sum_{\ell=1}^{N} \frac{1}{2} M_\ell (\dot{X}_\ell)^2 \tag{2.3}$$

is the kinetic energy, and

$$M_1 = M_2 = M_3 = m_1, \qquad \ldots \qquad M_{3N-2} = M_{3N-1} = M_{3N} = m_N \tag{2.4}$$

This is a special case of the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}_\ell}\right) - \frac{\partial L}{\partial X_\ell} = 0 \qquad \ell = 1, 2, \ldots 3N \tag{2.5}$$

where $L = T - V$ is the Lagrangian. Similar equations hold in any set of $3N$ "generalized" coördinates $\{q_k\}$, i.e.,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad k = 1, 2, \ldots 3N \tag{2.6}$$

2.1 The Springy Pendulum

As an example, consider a pointlike pendulum bob of mass m attached to the end of a spring of spring constant k and unstretched length b, moving in a plane under the influence of a constant gravitational field of magnitude g. Choose as generalized coördinates the (actual, instantaneous) length ℓ of the spring and the angle α between the spring and the vertical. These are something like polar coördinates, so the kinetic energy of the bob is

$$T = \frac{1}{2} m \dot{\ell}^2 + \frac{1}{2} m \ell^2 \dot{\alpha}^2 \tag{2.7}$$

There are two sources of potential energy: gravity and the spring. The gravitational potential energy can be worked out with a little trigonometry as $-mg\ell\cos\alpha$, while the potential energy in the spring is $\frac{1}{2}k(\ell-b)^2$. So the Lagrangian is

$$L = \frac{1}{2}m\dot{\ell}^2 + \frac{1}{2}m\ell^2\dot{\alpha}^2 + mg\ell\cos\alpha - \frac{1}{2}k(\ell-b)^2 \tag{2.8}$$

The relevant derivatives are

$$\frac{\partial L}{\partial \ell} = m\ell\dot{\alpha}^2 + mg\cos\alpha - k(\ell-b) \tag{2.9a}$$

$$\frac{\partial L}{\partial \alpha} = -mg\sin\alpha \tag{2.9b}$$

$$\frac{\partial L}{\partial \dot{\ell}} = m\dot{\ell} \tag{2.9c}$$

$$\frac{\partial L}{\partial \dot{\alpha}} = m\ell^2\dot{\alpha} \tag{2.9d}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\ell}}\right) = m\ddot{\ell} \tag{2.9e}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}}\right) = m\ell^2\ddot{\alpha} + 2m\ell\dot{\ell}\dot{\alpha} \tag{2.9f}$$

which makes the equations of motion

$$m\ddot{\ell} = m\ell\dot{\alpha}^2 + mg\cos\alpha - k(\ell-b) \tag{2.10a}$$

$$m\ell^2\ddot{\alpha} + 2m\ell\dot{\ell}\dot{\alpha} = -mg\ell\sin\alpha \tag{2.10b}$$

Note that, if we define a unit vector $\hat{\ell}$ in the direction of increasing ℓ (i.e., radially outward), the force due to the spring is

$$\vec{F}_{\text{spring}} = -k(\ell-b)\hat{\ell} \tag{2.11}$$

and this is reflected by a $-k(\ell-b)$ which appears in the $\ddot{\ell}$ equation.

What if there were a rod rather than a spring, so that the ℓ coördinate were fixed to be b? Physically, the rod would provide an additional radial force (pushing or tension) which was just what was needed to keep ℓ constant.

The key thing about a constraining force (e.g., tension, normal force, static friction) is that we don't know it *a priori* at each instant or at each point in space. It has to be just enough to keep the particle's motion consistent with the constraint.

If we write the tension in the rod as λ (which could be negative if the rod is pushing rather than pulling on the pendulum bob), the equations of motion are

$$m\ddot{\ell} = m\ell\dot{\alpha}^2 + mg\cos\alpha - \lambda \tag{2.12a}$$

$$m\ell^2\ddot{\alpha} + 2m\ell\dot{\ell}\dot{\alpha} = -mg\ell\sin\alpha \tag{2.12b}$$

And additionally there is a constraint

$$\ell = b \tag{2.12c}$$

The tension λ at any time is only worked out after the fact from the actual trajectory.

There are two ways to obtain the equations of motion (2.12) from a Lagrangian formalism:

1. Since one of the coördinates is a constant, we can make the substitutions $\ell = b$, $\dot{\ell} = 0$, and $\ddot{\ell} = 0$ into (2.12b) and get

$$mb^2\ddot{\alpha} = -mgb\sin\alpha \tag{2.13}$$

which are Lagrange's equations if we start with the reduced Lagrangian

$$L^{\text{red}}(\alpha, \dot{\alpha}, t) = \frac{1}{2}mb^2\dot{\alpha}^2 + mgb\cos\alpha \tag{2.14}$$

In the reduced Lagrangian, the constraint is satisfied and only the unconstrained "degree of freedom" α is treated as a generalized coördinate.

2. We can instead treat ℓ, α, *and* λ as the variables and look for a Lagrangian $\widetilde{L}(\ell, \alpha, \lambda, \dot{\ell}, \dot{\alpha}, \dot{\lambda}, t)$ which gives the equations of motion (2.12).

If we note that

$$-\lambda = \frac{\partial}{\partial \ell}(-\ell\lambda) \tag{2.15}$$

we see that

$$m\ddot{\ell} - m\ell\dot{\alpha}^2 = mg\cos\alpha - \lambda = \frac{\partial}{\partial \ell}\left(mg\ell\cos\alpha - \lambda\ell\right) \tag{2.16}$$

so we would get the right equation for $\ddot{\ell}$ if we add $-\lambda\ell$ to the Lagrangian:

$$\widetilde{L} \stackrel{?}{=} \frac{1}{2}m\dot{\ell}^2 + \frac{1}{2}m\ell^2\dot{\alpha}^2 + mg\ell\cos\alpha - \lambda\ell \tag{2.17}$$

That's not quite right, though, since if we take the derivatives associated with the tension, we get

$$\frac{\partial \widetilde{L}}{\partial \lambda} = -\ell \tag{2.18a}$$

$$\frac{\partial \widetilde{L}}{\partial \dot{\lambda}} = 0 \tag{2.18b}$$

$$\frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \dot{\lambda}}\right) = 0 \tag{2.18c}$$

since the constraint is $\ell = b$, not $\ell = 0$. But if we add $\lambda(b - \ell)$ instead of $-\lambda\ell$, we don't change $\frac{\partial \widetilde{L}}{\partial \lambda}$ and we do get

$$\frac{\partial \widetilde{L}}{\partial \lambda} = b - \ell = 0 = \frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \dot{\ell}}\right) \tag{2.19}$$

so the constraint is just the Lagrange equation associated with the unknown tension λ.

So we can describe the pendulum with the Lagrangian

$$\widetilde{L} = \underbrace{\frac{1}{2}m\dot{\ell}^2 + \frac{1}{2}m\ell^2\dot{\alpha}^2}_{T} + \underbrace{mg\ell\cos\alpha}_{-V} + \underbrace{\lambda}_{\substack{\text{Lagrange}\\\text{multiplier}}}\underbrace{(b - \ell)}_{\text{constraint}} \tag{2.20}$$

In Cartesian coördinates (centered at the support point of the pendulum) the constraint is

$$b - \sqrt{x^2 + y^2} = 0 \tag{2.21}$$

so we can't just eliminate one coördinate by setting it to a constant. In this case, the constraining force has components in both directions. Since

$$\cos\alpha = -\frac{y}{\sqrt{x^2 + y^2}} \tag{2.22a}$$

$$\sin\alpha = \frac{x}{\sqrt{x^2 + y^2}} \tag{2.22b}$$

if we call the tension $\vec{T}$ and let λ be $\pm\left|\vec{T}\right|$ (with the sign depending on whether $\vec{T}$ is a pull or a push), then

$$T_x = -\lambda\sin\alpha = -\frac{x}{x^2 + y^2} = \lambda\frac{\partial}{\partial x}\left(b - \sqrt{x^2 + y^2}\right) \tag{2.23a}$$

$$T_y = \lambda\cos\alpha = -\frac{y}{x^2 + y^2} = \lambda\frac{\partial}{\partial y}\left(b - \sqrt{x^2 + y^2}\right) \tag{2.23b}$$

so, again we get the right equations by adding λ times the constraint to the Lagrangian:

$$\widetilde{L}(x, y, \lambda, \dot{x}, \dot{y}, t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + mgy + \lambda\left(b - \sqrt{x^2 + y^2}\right) \tag{2.24}$$

The circle defined by $\ell = b$ is called a *surface of constraint*. The tension in the rod is directed radially, i.e., perpendicular to that circle. In polar coördinates, we can write the constraint as $h(r, \phi) = b - r = 0$ and note that $\vec{\nabla}h = \hat{r}$ is perpendicular to the surface of the constraint, so the constraining force $-\lambda\hat{r}$ is parallel to $\vec{\nabla}h$.

2.2 Constrained Systems in General

Think about the general case with $3N$ Cartesian cördinates for N particles and require these to satisfy c constraints

$$h_1(\{X_\ell\}) = 0 \tag{2.25a}$$

$$h_2(\{X_\ell\}) = 0 \tag{2.25b}$$

$$\vdots$$

$$h_c(\{X_\ell\}) = 0 \tag{2.25c}$$

Each constraint has an associated constraining force which acts on each particle. For one particle, the force corresponding to the jth constraint, which might be called $\vec{F}_j$, will be perpendicular to the surface $h_j(\vec{r}) = 0$, and thus be parallel to $\vec{\nabla}h_j$. so we could write it as $\vec{F}_j^{\text{const}} = \lambda_j\vec{\nabla}h_j$. For N particles, the force associated with the jth constraint, acting on the ith particle, will be

$$\vec{F}_{ji}^{\text{const}} = \lambda_j\vec{\nabla}_i h_j \tag{2.26}$$

In our current notation for Cartesian coördinates, this makes the equations of motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{X}_\ell}\right) = M_\ell \ddot{X}_\ell = -\frac{\partial V}{\partial X_\ell} + \sum_{j=1}^{c} \lambda_j \frac{\partial h_j}{\partial X_\ell} \tag{2.27}$$

Now, although we know the directions of the constraining forces, their strengths $\{\lambda_j | j = 1 \ldots c\}$ are unknown, which means there are $3N + c$ unknowns

$$X_1, X_2, \ldots, X_{3N}, \lambda_1, \lambda_2, \ldots, \lambda_c \tag{2.28}$$

We have a total of $3N + c$ equations of motion: the $3N$ dynamical equations (2.27) plus the c constraints (2.25). Now, we can turn the unconstrained differential equations into the constrained ones by replacing V with $V - \sum_{j=1}^{c} \lambda_j h_j$ so if we define a *modified Lagrangian*

$$\widetilde{L} = L + \sum_{j=1}^{c} \lambda_j h_j = T - V + \sum_{j=1}^{c} \lambda_j h_j \tag{2.29}$$

we will get the right equations of motion. First, if we hold all the $\{\lambda_j\}$ constant in the derivatives,

$$\frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \dot{X}_\ell}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}_\ell}\right) = M \ddot{X}_\ell = -\frac{\partial V}{\partial X_\ell} + \sum_{j=1}^{c} \lambda_j \frac{\partial h_j}{\partial X_\ell} = \frac{\partial \widetilde{L}}{\partial X_\ell} \tag{2.30}$$

On the other hand if we take a partial derivative with respect to each λ_j, holding the $\{X_\ell\}$ and $\{\dot{X}_\ell\}$ constant,

$$\frac{\partial \widetilde{L}}{\partial X_\ell} = \sum_{j'=1}^{c} \underbrace{\frac{\partial \lambda_{j'}}{\partial \lambda_j}}_{\substack{0 \text{ if } j' \neq j \\ 1 \text{ if } j' = j}} h_{j'}(\{X_\ell\}) = h_j(\{X_\ell\}) = 0 = \frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \dot{\lambda}_j}\right) \tag{2.31}$$

So the $3N + c$ equations of motion are

$$\frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \dot{X}_\ell}\right) - \frac{\partial \widetilde{L}}{\partial X_\ell} = 0 \qquad \ell = 1, 2, \ldots 3N \tag{2.32a}$$

$$\frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \dot{\lambda}_k}\right) - \frac{\partial \widetilde{L}}{\partial \lambda_k} = 0 \qquad k = 1, 2, \ldots c \tag{2.32b}$$

This is called the method of *Lagrange undetermined multipliers*. The modified Lagrangian is a function of $3N{+}c$ "coördinates" $\{X_1, X_2, \ldots, X_{3N}, \lambda_1, \lambda_2, \ldots, \lambda_c\}$ and their time derivatives (it happens to be independent of $\{\dot{\lambda}_j\}$)

$$\widetilde{L}(\{X_\ell\}, \{\lambda_j\}, \{\dot{X}_\ell\}, t) = L(\{X_\ell\}, \{\lambda_j\}, \{\dot{X}_\ell\}, t) + \sum_{j=1}^{c} \lambda_j h_j(\{X_\ell\}) \tag{2.33}$$

Now, we could replace these $3N+c$ coördinates with $3N+c$ different ones, and the equations of motion would still come from applying Lagrange's equations to $\widetilde{L}$.

Suppose we manage to choose those in such a way that c of them are just the constraints, c of them are still the Lagrange multipliers, and the remaining

$$3N - c =: f \tag{2.34}$$

are something else. Then f is the number of "degrees of freedom" which is just the number of unconstrained "directions" the system can move in. the old and new coördinates would be

Old:	X_1	X_2	$\ldots$	X_f	X_{f+1}	$\ldots$	X_{f+c}	λ_1	$\ldots$	λ_c
New:	q_1	q_2	$\ldots$	q_f	a_1	$\ldots$	a_c	λ_1	$\ldots$	λ_c

Substituting the inverse transformations $X_\ell = X_\ell(\{q_k|k=1\ldots f\},\{a_j|j=1\ldots c\})$ into the modified Lagrangian, we'd get the form

$$\widetilde{L}(\{q_k\},\{a_j\},\{\lambda_j\},\{\dot{q}_k\},\{\dot{a}_j\},t) = L(\{q_k\},\{a_j\},\{\dot{q}_k\},\{\dot{a}_j\},t) + \sum_{j=1}^{c} \lambda_j a_j \tag{2.35}$$

The Lagrange equations are then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} = \frac{\partial \widetilde{L}}{\partial q_k} \tag{2.36a}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{a}_j}\right) = \frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \dot{a}_j}\right) = \frac{\partial L}{\partial a_j} = \frac{\partial \widetilde{L}}{\partial a_j} + \lambda_j \tag{2.36b}$$

$$0 = \frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \dot{\lambda}_j}\right) = \frac{\partial L}{\partial \lambda_j} = a_j \tag{2.36c}$$

- Now, the last set of Lagrange equations tell us we can impose the constraints and set $a_j = 0$ for $j = 1\ldots c$ in the other two.
- The second set is only really needed if we care about the constraining forces, i.e., the Lagrange multipliers λ_j.
- The first set tells us

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k}\right]_{\substack{a_1=0\\ \vdots\\ a_c=0}} = 0 \qquad k = 1\ldots f \tag{2.37}$$

If we define the *reduced Lagrangian*

$$L^{\text{red}}(\{q_k\},\{\dot{q}_k\},t) = L(\{q_k\},\{a_j=0\},\{\dot{q}_k\},\{\dot{a}_j=0\},t) \tag{2.38}$$

then the Lagrange equations are equivalent to

$$\frac{d}{dt}\left(\frac{\partial L^{\text{red}}}{\partial \dot{q}_k}\right) - \frac{\partial L^{\text{red}}}{\partial q_k} = 0 \qquad k = 1\ldots f \tag{2.39}$$

This means that if a system is constrained to have only f degrees of freedom, we can formulate the Lagrangian as a function of the f generalized coördinates and apply the usual Lagrange equations. as long as we don't care about the constraining forces.

2.3 Example: The Atwood Machine

As an example of how to apply the method of Lagrange multipliers to a constrained system, consider the Atwood machine. This consists of two blocks, of masses m_1 and m_2, at opposite ends of a massless rope of (fixed) length ℓ, hung over a massless, frictionless pulley, subject to a uniform gravitational field of strength g. This is illustrated in Symon's Fig. 9.5. Symon considers this system in the reduced Lagrangian approach in Section 9.5.

We are looking for the acceleration of each block, and possibly also the tension in the rope.

2.3.1 Lagrange Multiplier Approach

Here we will use the modified Lagrangian approach, with a Lagrange multiplier corresponding to the unknown constraining force, here provided by the tension in the rope.

We define generalized coördinates x_1 and x_2, which are both supposed to be positive, and refer to the distance of each block *below* the pulley. The constraint here is that the fixed length of the rope is $\ell = x_1 + x_2$. To construct the modified Lagrangian, we need the kinetic energy

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2 \tag{2.40}$$

and the potential energy

$$V = -m_1 g x_1 - m_2 g x_2 \tag{2.41}$$

(since the height of each block *above* the pulley is $-x_1$ or $-x_2$). The modified Lagrangian is then

$$\widetilde{L}(x_1, x_2, \lambda, \dot{x}_1, \dot{y}_1, t) = T - V + \lambda(\ell - x_1 - x_2) = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + m_1 g x_1 + m_2 g x_2 + \lambda(\ell - x_1 - x_2) \tag{2.42}$$

and the equations of motion are

$$m_1\ddot{x}_1 = \frac{d}{dt}\left(\frac{\partial\widetilde{L}}{\partial\dot{x}_1}\right) = \frac{\partial L}{\partial x_1} = m_1 g - \lambda \tag{2.43a}$$

$$m_1\ddot{x}_2 = \frac{d}{dt}\left(\frac{\partial\widetilde{L}}{\partial\dot{x}_2}\right) = \frac{\partial L}{\partial x_2} = m_2 g - \lambda \tag{2.43b}$$

$$0 = \frac{d}{dt}\left(\frac{\partial\widetilde{L}}{\partial\dot{\lambda}}\right) = \frac{\partial L}{\partial \lambda} = \ell - x_1 - x_2 \tag{2.43c}$$

Since a positive tension will produce a force which tends to try to decrease x_1 and x_2, the tension in the rope is $\lambda > 0$.

To get separate equations for $\ddot{x}_1$ and $\ddot{x}_2$, we take time derivatives of the constraint (2.43c) to say

$$\dot{x}_1 + \dot{x}_2 = 0 \tag{2.44a}$$

$$\ddot{x}_1 + \ddot{x}_2 = 0 \tag{2.44b}$$

and so we can substitute $\ddot{x}_2 = -\ddot{x}_1$. We might also need $x_2 = \ell - x_1$ in a more general problem, but here the equations of motion turn out to be independent of x_2.

To eliminate λ from (2.43a) we solve (2.43b) for

$$-\lambda = m_2(\ddot{x}_2 - g) = m_2(-\ddot{x}_1 - g) \tag{2.45}$$

and then substitute in to get

$$m_1\ddot{x}_1 = m_1 g - m_2\ddot{x}_1 - m_2 g \tag{2.46}$$

which can be solved for

$$\ddot{x}_1 = \frac{m_1 - m_2}{m_1 + m_2}\, g \tag{2.47}$$

and then

$$\ddot{x}_2 = -\ddot{x}_1 = \frac{m_2 - m_1}{m_2 + m_1}\, g \tag{2.48}$$

and finally

$$\lambda = -m_1(\ddot{x}_1 + g) = m_1 g\left(\frac{-m_1 + m_2 + m_1 + m_2}{m_1 + m_2}\right) = \frac{2m_1 m_2}{m_1 + m_2} g \tag{2.49}$$

which is the tension in the rope.

2.3.2 Reduced Lagrangian Approach

The other way to handle this problem is to make the replacements $x_2 = \ell - x_1$ and $\dot{x}_2 = -\dot{x}_1$ at the level of the Lagrangian, and obtain the reduced Lagrangian with one degree of freedom:

$$L(x_1, \dot{x}_1, t) = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2(-\dot{x}_1)^2 + m_1 g x_1 + m_2 g(\ell - x_1) = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + (m_1 - m_2) g x_1 + m_2 g \ell \tag{2.50}$$

The equation of motion is then

$$(m_1 + m_2)\ddot{x}_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = \frac{\partial L}{\partial x_1} = (m_1 - m_2) g \tag{2.51}$$

or

$$\ddot{x}_1 = \frac{m_1 - m_2}{m_1 + m_2}\, g \tag{2.52}$$

It's a lot simpler to get the equation of motion from this approach, but we don't get the value of the tension.

3 Velocity-Dependent Potentials

So far we've started from the Newtonian picture and seen how to derive a Lagrangian for each class of problem. Once we'd justified the Lagrangian approach, we were able to construct the Lagrangian in generalized coördinates directly. Now, let's go a step further and start with a Lagrangian, and see what sort of systems it describes. The Lagrangian is, in Cartesian coördinates for a single particle,

$$L(\vec{r},\dot{\vec{r}},t) = \frac{1}{2}m\left|\dot{\vec{r}}\right|^2 - Q\varphi(\vec{r},t) + Q\dot{\vec{r}}\cdot\vec{A}(\vec{r},t) \tag{3.1}$$

where Q is a constant, $\varphi(\vec{r},t)$ is a scalar field, and $\vec{A}(\vec{r},t)$ is a vector field. Note that

$$Q\varphi(\vec{r},t) - Q\dot{\vec{r}}\cdot\vec{A}(\vec{r},t) \tag{3.2}$$

plays the role of the potential energy V, but now it depends on $\dot{\vec{r}}$ as well as $\vec{r}$. This sort of Lagrangian is associated with a "velocity-dependent potential".

Writing the Lagrangian out explicitly,

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2 - Q\varphi + Q\dot{x}A_x + Q\dot{y}A_y + Q\dot{z}A_z \tag{3.3}$$

where A_x, A_y, A_z, and φ are all functions of x, y, z, and t.

Let's focus on one Lagrange equation, the one corresponding to the x coördinate. The relevant derivatives are

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + QA_x \tag{3.4}$$

and then, using the chain rule to evaluate the total time derivative,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} + Q\frac{dA_x}{dt} = m\ddot{x} + Q\frac{\partial A_x}{\partial x}\dot{x} + Q\frac{\partial A_x}{\partial y}\dot{y} + Q\frac{\partial A_x}{\partial z}\dot{z} + Q\frac{\partial A_x}{\partial t} \tag{3.5}$$

and finally, because the Lagrangian depends on x through the fields $\vec{A}$ and φ,

$$\frac{\partial L}{\partial x} = -Q\frac{\partial \varphi}{\partial x} + Q\dot{x}\frac{A_x}{\partial x} + Q\dot{y}\frac{\partial A_y}{\partial x} + Q\dot{z}\frac{\partial A_z}{\partial x} \tag{3.6}$$

Now, the Lagrange equation is

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} \\ &= m\ddot{x} + \cancel{Q\dot{x}\frac{\partial A_x}{\partial x}} + Q\dot{y}\frac{\partial A_x}{\partial y} + Q\dot{z}\frac{\partial A_x}{\partial z} + Q\frac{\partial A_x}{\partial t} + Q\frac{\partial \varphi}{\partial x} - \cancel{Q\dot{x}\frac{A_x}{\partial x}} - Q\dot{y}\frac{\partial A_y}{\partial x} - Q\dot{z}\frac{\partial A_z}{\partial x} \\ &= m\ddot{x} + Q\left(\frac{\partial A_x}{\partial t} + \frac{\partial \varphi}{\partial x}\right) + Q\dot{y}\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right) + Q\dot{z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \end{aligned} \tag{3.7}$$

which we can solve for

$$m\ddot{x} = Q\left(\left[-\frac{\partial \varphi}{\partial x} - \frac{\partial A_x}{\partial t}\right] + \dot{y}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] - \dot{z}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\right) \tag{3.8}$$

The first expression in brackets is the x component of $-\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}$, the second expression in brackets is the z component of $\vec{\nabla} \times \vec{A}$, and the third expression in brackets is the y

component of $\vec{\nabla} \times \vec{A}$. But if we interpret φ as the scalar potential and $\vec{A}$ as the vector potential of electrodynamics, these are just components of the electric and magnetic fields

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t} \tag{3.9a}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{3.9b}$$

so

$$m\ddot{x} = Q\big(E_x + \underbrace{\dot{y}B_z - \dot{z}B_y}_{x \text{ compt of } \dot{\vec{r}} \times \vec{B}}\big) = \hat{x} \cdot \left[Q\left(\vec{E} + \dot{\vec{r}} \times \vec{B}\right)\right] \tag{3.10}$$

The calculations for the y and z components of the acceleration are similar and the Lagrange equations arising from

$$L = \frac{1}{2}m\left|\dot{\vec{r}}\right|^2 - Q\varphi + Q\dot{\vec{r}}\cdot\vec{A} \tag{3.11}$$

are just the components of the Lorentz force law

$$m\ddot{\vec{r}} = Q\left(\vec{E} + \dot{\vec{r}} \times \vec{B}\right) \tag{3.12}$$

More on this example in the future.

4 Conservation Laws

4.1 Conservation of Momentum and Ignorable Coördinates

Recall conservation of momentum in Newtonian physics:

$$\frac{d\vec{p}}{dt} = m\ddot{\vec{r}} = \vec{F} = -\vec{\nabla}V \tag{4.1}$$

$$p_x = m\dot{x} = \text{constant} \qquad \text{if } \frac{\partial V}{\partial x} = 0 \tag{4.2a}$$

$$p_y = m\dot{y} = \text{constant} \qquad \text{if } \frac{\partial V}{\partial y} = 0 \tag{4.2b}$$

$$p_z = m\dot{z} = \text{constant} \qquad \text{if } \frac{\partial V}{\partial z} = 0 \tag{4.2c}$$

If V is independent of one of the Cartesian coördinates, the corresponding component of the momentum is a constant of the motion.

In Lagrangian mechanics, with generalized coördinates,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial q_k} \tag{4.3}$$

so if L is independent if one of the coördinates q_k, the corresponding quantity $\frac{\partial L}{\partial \dot{q}_k}$ is a constant. We call this the *generalized momentum*

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \tag{4.4}$$

(This definition applies in general, even if $\frac{\partial L}{\partial q_k} \neq 0$.) The generalized momentum is sometimes also called the *canonical momentum*. We say that a given p_k is *canonically conjugate* to the corresponding q_k.

For Cartesian coördinates, this is just one of the components of the momentum of one of the particles.

$$p_{xi} = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i \tag{4.5a}$$

$$p_{yi} = \frac{\partial L}{\partial \dot{y}_i} = m\dot{y}_i \tag{4.5b}$$

$$p_{zi} = \frac{\partial L}{\partial \dot{z}_i} = m\dot{z}_i \tag{4.5c}$$

(i is particle label)

As an example of the definition of generalized momenta for non-Cartesian coördinates, consider the Lagrangian for a single particle in two dimensions, described in polar coördinates:

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - V(r,\phi) \tag{4.6}$$

the generalized momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \tag{4.7a}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} \tag{4.7b}$$

Note that these are *not* just the components of the usual momentum vector

$$\vec{p} = m\dot{r}\,\hat{r} + mr\dot{\phi}\,\hat{\phi} \tag{4.8}$$

In particular,

$$p_\phi \neq \hat{\phi} \cdot \vec{p} \tag{4.9}$$

(For one thing, the two quantities have different units!) Thus p_ϕ refers to something different than it did last semester. There's not really an ideal way around this notational hassle. We could write $p_{\hat{\phi}}$ for $\hat{\phi} \cdot p_\phi$, but we wouldn't want to have been doing that all along. Another convention would be to call the generalized momenta $\{\pi_k\}$ rather than $\{p_k\}$, but that's also potentially confusing.

In fact, there is a physical interpretation for p_ϕ: it's just the z component of the orbital angular momentum of the particle:

$$p_\phi = mr^2\dot{\phi} = L_z = \hat{z} \cdot [\vec{r} \times (m\dot{\vec{r}})] \tag{4.10}$$

Another example where the generalized momenta are not the components of the ordinary momentum vector is in the electromagnetic Lagrangian we consdidered last time:

$$L = \frac{1}{2}m\left|\dot{\vec{r}}\right|^2 - Q\varphi + Q\dot{\vec{r}} \cdot \vec{A} \tag{4.11}$$

In Cartesian coördinates, we get, for example

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + QA_x \tag{4.12}$$

and similarly for the y and z momenta. Here the canonical momenta can even be combined into a vector

$$p_x\hat{x} + p_y\hat{y} + p_z\hat{z} = m\dot{\vec{r}} + Q\vec{A} \tag{4.13}$$

Whether to call this "canonical momentum vector" $\vec{p}$ (or, for example, $\vec{\pi}$) is again a matter of notational preference.

Returning to the original topic of conservation laws, if L is independent of q_k, then p_k is a constant of the motion (its time derivative vanishes as a result of Lagrange's equations), and q_k is called an *ignorable coördinate.*

So if the Lagrangian is independent of y (translationally invariant), the component of momentum in the y direction vanishes. If it's independent of ϕ (rotationally invariant about an axis) the component of angular momentum along that axis vanishes. In general in Physics, a symmetry is associated with a conservation law.

Note that it's L that needs to be independent of q_k, not just V. For example, in polar coördinates, the Lagrange equation associated with the r coördinate tells us

$$\frac{dp_r}{dt} = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} = mr\dot{\phi}^2 + \frac{\partial V}{\partial r} \tag{4.14}$$

which does *not* vanish even if the potential V is independent of r.

4.2 Conservation of Energy and Definition of the Hamiltonian

Recall conservation of energy, $T + V = E =$ constant, i.e., $\frac{d}{dt}(T+V) = 0$. How did that come about in the simplest situation, one particle in one dimension with $T = \frac{1}{2}m\dot{x}^2$?

$$\frac{d}{dt}(T+V) = \frac{dT}{dt} + \frac{dV}{dt} \tag{4.15}$$

where

$$\frac{dT}{dt} = \frac{dT}{d\dot{x}}\frac{d\dot{x}}{dt} = m\dot{x}\ddot{x} = \dot{x}F(x) = \dot{x}\left(-\frac{dV}{dx}\right) \tag{4.16}$$

(using Newton's 2nd law) while

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = \frac{dV}{dx}\dot{x} \tag{4.17}$$

so

$$\frac{dT}{dt} = -\frac{dV}{dt} \tag{4.18}$$

Note this only works because V is independent of time. If we had $V(x,t)$ with explicit time dependence as well as that implicit in the x, we'd get

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial t}\frac{dt}{dt} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial t} \tag{4.19}$$

Now $L = T - V$; can we use Lagrange eqns to derive conservation of energy? Use chain rule to find time derivative of $L(t) = L(\{q_k(t)\}, \{\dot{q}_k(t)\}, t)$

$$\begin{aligned}\frac{dL}{dt} &= \sum_{k=1}^{f}\left(\frac{\partial L}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial L}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt}\right) + \frac{\partial L}{\partial t} = \sum_{k=1}^{f}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k\right) + \frac{\partial L}{\partial t} \\ &= \frac{d}{dt}\left(\sum_{k=1}^{f}\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right) + \frac{\partial L}{\partial t}\end{aligned} \tag{4.20}$$

Or, putting all the total derivatives on one side,

$$\frac{d}{dt}\left(\sum_{k=1}^{f}\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L\right) = -\frac{\partial L}{\partial t} \tag{4.21}$$

If the Lagrangian has no explicit time dependence ($\frac{\partial L}{\partial t} = 0$), then

$$H = \sum_{k=1}^{f}\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L \tag{4.22}$$

is a constant of the motion. H is called the Hamiltonian, and is often but not always equal to the total energy $T + V$.

4.2.1 When is the Hamiltonian the Total Energy?

If you start from the Cartesian definition

$$T = \sum_{\ell=1}^{3N}\frac{1}{2}M_\ell \dot{X}_\ell^2 \tag{4.23}$$

and substitute in

$$\dot{X}_\ell = \sum_{k=1}^{f}\frac{\partial X_\ell}{\partial q_k}\dot{q}_k + \frac{\partial X_\ell}{\partial t} \tag{4.24}$$

which we found in Sec. (1.1.5)[3], you can see that $T(\{q_k\}, \{\dot{q}_k\}, t)$ is in general made up of pieces which are quadratic, linear and independent of the generalized velocities $\{\dot{q}_k\}$:

$$T = \underbrace{\sum_{k'=1}^{f}\sum_{k''=1}^{f}\frac{1}{2}A_{k'k''}(\{q_k\}, t)\dot{q}_{k'}\dot{q}_{k''}}_{T_2(\{q_k\},\{\dot{q}_k\},t)} + \underbrace{\sum_{k'=1}^{f}B_{k'k''}(\{q_k\}, t)\dot{q}_{k'}}_{T_1(\{q_k\},\{\dot{q}_k\},t)} + T_0(\{q_k\}, t) \tag{4.25}$$

[3]Technically speaking, we only did this for completely unconstrained systems, but the introduction of constraints to reduce the Lagrangian doesn't change things; the expression given here is just the Cartesian velocity in terms of the unconstrained degrees of freedom, once the constraints are imposed.

We can show that

$$\sum_{k=1}^{f} \frac{\partial T_2}{\partial \dot{q}_k} \dot{q}_k = 2T_2 \tag{4.26a}$$

$$\sum_{k=1}^{f} \frac{\partial T_1}{\partial \dot{q}_k} \dot{q}_k = T_1 \tag{4.26b}$$

$$\sum_{k=1}^{f} \frac{\partial T_0}{\partial \dot{q}_k} \dot{q}_k = 0 \tag{4.26c}$$

So in general

$$H = 2T_2 + T_1 - (T_2 + T_1 + T_0 - V) = T_2 - T_0 - V \tag{4.27}$$

So the Hamiltonian is the same as the total energy $T + V$ if the kinetic energy is purely quadratic, $T_1 = 0 = T_0$ so that $T = T_2$.

4.2.2 Examples

Examples where H is or is not the total energy.

5 Hamiltonian Mechanics

So far: system w/f degrees of freedom described by f generalized coördinates $\{q_k | k = 1 \ldots f\}$

Construct a Lagrangian $L(\{q_k\}, \{\dot{q}_k\}, t)$ which is usually $T - V$.

The actual trajectory $\{q_k(t)\}$ satisfies

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} \qquad \text{for all } k = 1 \ldots f \tag{5.1}$$

Note, these are second-order differential equations because $\frac{\partial L}{\partial \dot{q}_k}$ will usually contain time derivatives $\{\dot{q}_{k'}\}$.

Last week you saw that $p_k = \frac{\partial L}{\partial \dot{q}_k}$ was a constant of the motion if $\left(\frac{\partial L}{\partial q_k}\right)_{\{q_{k' \neq k}\}, \{\dot{q}_{k'}\}, t} = 0$, because $\frac{dp_k}{dt} = \frac{\partial L}{\partial q_k}$.

You also saw that "X"$= \sum_{k=1}^{f} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L$ was conserved if L had no explicit time dependence, and this "X" was often the total energy $T + V$. This quantity is called the Hamiltonian, and we use the symbol H to refer to it.

We thought of p_k as some function of the $\{q_{k'}\}$ & $\{\dot{q}_{k'}\}$ but since we're taking a total derivative, it could also be thought of as some function of time which is determined by the trajectory, so $\dot{p}_k = \frac{\partial L}{\partial q_k}$ as a consequence of the Lagrange equations.

This is an interesting situation:

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \tag{5.2a}$$

$$\dot{p}_k = \frac{\partial L}{\partial q_k} \tag{5.2b}$$

A good way to summarize partial derivatives is to think about the infinitesimal change in a function associated with infinitesimal changes in its arguments. So for $f(x, y)$, we have

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \tag{5.3}$$

This is a distillation of the chain rule (with a hint of implicit differentiation thrown in).

Apply this to $L(\{q_k\}, \{\dot{q}_k\}, t)$:

$$\begin{aligned} dL &= \frac{\partial L}{\partial q_1}\,dq_1 + \frac{\partial L}{\partial q_2}\,dq_2 + \ldots + \frac{\partial L}{\partial q_f}\,dq_f + \frac{\partial L}{\partial \dot{q}_1}\,d\dot{q}_1 + \frac{\partial L}{\partial \dot{q}_2}\,d\dot{q}_2 + \ldots + \frac{\partial L}{\partial \dot{q}_f}\,d\dot{q}_f + \frac{\partial L}{\partial t}\,dt \\ &= \sum_{k=1}^{f}\left(\frac{\partial L}{\partial q_k}\,dq_k + \frac{\partial L}{\partial \dot{q}_k}\,d\dot{q}_k\right) + \frac{\partial L}{\partial t}\,dt = \sum_{k=1}^{f}(\dot{p}_k\,dq_k + p_k\,d\dot{q}_k) + \frac{\partial L}{\partial t}\,dt \end{aligned} \tag{5.4}$$

Now think about the definition of the Hamiltonian

$$H = \sum_{k=1}^{f} p_k \dot{q}_k - L \tag{5.5}$$

well,

$$\begin{aligned} dH &= \sum_{k=1}^{f}(\cancel{p_k\,d\dot{q}_k} + \dot{q}_k\,dp_k) - \sum_{k=1}^{f}(\dot{p}_k\,dq_k + \cancel{p_k\,d\dot{q}_k}) - \frac{\partial L}{\partial t}\,dt \\ &= \sum_{k=1}^{f}(\dot{q}_k\,dp_k - \dot{p}_k\,dq_k) - \frac{\partial L}{\partial t}\,dt \end{aligned} \tag{5.6}$$

The $d\dot{q}$ terms cancel. This means that it's much easier to think of H as a function of $\{q_k\}$, $\{p_k\}$ and t rather than $\{q_k\}$, $\{\dot{q}_k\}$ and t. Its partial derivatives are

$$\left(\frac{\partial H}{\partial p_k}\right)_{\{q_{k'\neq k}\},\{p_{k'}\},t} = \dot{q}_k \tag{5.7a}$$

$$\left(\frac{\partial H}{\partial q_k}\right)_{\{q_{k'}\},\{p_{k'\neq k}\},t} = -\dot{p}_k \tag{5.7b}$$

$$\left(\frac{\partial H}{\partial t}\right)_{\{q_k\},\{p_k\}} = -\left(\frac{\partial L}{\partial t}\right)_{\{q_k\},\{\dot{q}_k\}} \tag{5.7c}$$

The first two equations are called *Hamilton's equations* and contain the same information as the Lagrange equations.

What we have just performed is called a *Legendre transformation* from $L(\{q_k\}, \{\dot{q}_k\}, t)$ to $H(\{q_k\}, \{p_k\}, t)$:

- Construct $p_k(\{q_{k'}\}, \{\dot{q}_{k'}\}, t) = \frac{\partial L}{\partial p_k}$
- Invert to get $\dot{q}_k = \dot{q}_k(\{q_{k'}\}, \{p_{k'}\}, t)$
- $$H(\{q_k\}, \{p_k\}, t) = \sum_{k'=1}^{f} p_{k'}\dot{q}_k(\{q_{k'}\}, \{p_{k'}\}, t) - LL(\{q_k\}, \{\dot{q}_k(\{q_{k'}\}, \{p_{k'}\}, t)\}, t) \tag{5.8}$$

5.1 Examples

5.1.1 Cartesian Coördinates

For N particles in three dimensions, we have as usual

$$L = \sum_{\ell=1}^{3N} \frac{1}{2} M_\ell \dot{X}_\ell^2 - V(\{X_\ell\}) \tag{5.9}$$

The canonical momentum conjugate to a particular X_ℓ is

$$P_\ell = \frac{\partial L}{\partial \dot{X}_\ell} = M_\ell \dot{X}_\ell \tag{5.10}$$

which can be inverted to get

$$\dot{X}_\ell = \frac{P_\ell}{M_\ell} \tag{5.11}$$

and then we find the Hamiltonian

$$\begin{aligned} H &= \sum_{\ell=1}^{3N} P_\ell \dot{X}_\ell - \sum_{\ell=1}^{3N} \frac{1}{2} M_\ell \dot{X}_\ell^2 + V(\{X_\ell\}) \\ &= \sum_{\ell=1}^{3N} \frac{P_\ell^2}{M_\ell} - \sum_{\ell=1}^{3N} \frac{1}{2} M_\ell \left(\frac{P_\ell}{M_\ell}\right)^2 + V(\{X_\ell\}) \\ &= \sum_{\ell=1}^{3N} \frac{P_\ell^2}{2M_\ell} + V(\{X_\ell\}) \end{aligned} \tag{5.12}$$

which then has Hamilton equations of

$$\frac{\partial H}{\partial P_\ell} = \frac{P_\ell}{M_\ell} = \dot{X}_\ell \tag{5.13a}$$

$$\frac{\partial H}{\partial X_\ell} = \frac{\partial V}{\partial X_\ell} = -\dot{P}_\ell \tag{5.13b}$$

5.1.2 Polar Coördinates

In terms of polar coördinates r and ϕ, the Lagrangian is

$$L = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 - V(r, \phi) \tag{5.14}$$

The conjugate momenta are

$$p_r = m\dot{r} \tag{5.15a}$$

$$p_\phi = m r^2 \dot{\phi} \tag{5.15b}$$

which are, physically, the radial component of momentum and the angular momentum, respectively.

Inverting this gives

$$\dot{r} = \frac{p_r}{m} \tag{5.16a}$$

$$\dot{\phi} = \frac{p_\phi}{mr^2} \tag{5.16b}$$

Note the r dependence of $\dot{\phi}$.

The Hamiltonian is then

$$\begin{aligned} H &= p_r\dot{r} + p_\phi\dot{\phi} - \frac{1}{2}m\dot{r}^2 - \frac{1}{2}mr^2\dot{\phi}^2 + V(r,\phi) \\ &= \frac{p_r^2}{m} + \frac{p_\phi}{mr^2} - \frac{m}{2}\left(\frac{p_r}{m}\right)^2 - \frac{mr^2}{2}\left(\frac{p_\phi}{2mr^2}\right)^2 + V(r,\phi) \\ &= \frac{p_r^2}{2m} + \frac{p_\phi}{2mr^2} + V(r,\phi) \end{aligned} \tag{5.17}$$

and Hamilton's equations are

$$\frac{\partial H}{\partial p_r} = \frac{p_r}{m} = \dot{r} \tag{5.18a}$$

$$\frac{\partial H}{\partial r} = -\frac{p_\phi^2}{mr^3} + \frac{\partial V}{\partial r} = -\dot{p}_r \tag{5.18b}$$

$$\frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2} = \dot{\phi} \tag{5.18c}$$

$$\frac{\partial H}{\partial r} = \frac{\partial V}{\partial \phi} = -\dot{p}_\phi \tag{5.18d}$$

Note that if V is $V(r)$, as in the case of a central force, ϕ is an "ignorable coördinate" and p_ϕ is a constant.

5.2 More Features of the Hamiltonian

Last time, we constructed a Hamiltonian out of a Lagrangian and then showed Lagrange's equations implied Hamilton's equations. To stress the completeness of the Hamiltonian picture, let's state the formulation without reference to the Lagrangian picture:

Given a Hamiltonian $H(\{q_k\},\{p_k\},t)$, Hamilton's equations are

$$\frac{\partial H}{\partial q_k} = -\dot{p}_k \qquad k = 1,\dots,f \tag{5.19a}$$

$$\frac{\partial H}{\partial p_k} = \dot{q}_k \qquad k = 1,\dots,f \tag{5.19b}$$

What is this Hamiltonian? Starting from the Lagrangian $L(\{q_k\},\{\dot{q}_k\},t)$, construct $p_k = \frac{\partial L}{\partial p_k} = p_k(\{q_{k'}\},\{\dot{q}_{k'}\},t)$ and $H = \sum_{k=1}^{f} p_k\dot{q}_k - L$, using the inverse $\dot{q}_k = \dot{q}_k(\{q_{k'}\},\{p_{k'}\},t)$ to write H as a function of the ps and qs but not the $\dot{q}$s.

The Hamiltonian is "often" the total energy $E = T + V$. Basically, they're the same whenever V is a function only of the $\{q_k\}$ and T is purely quadratic in the $\{\dot{q}_k\}$. Look at an example for $f = 1$ to make this explicit without worrying about subscripts …

From the kinetic energy

$$T = \frac{1}{2} a(q,t)\,\dot{q}^2 + b(q,t)\,\dot{q} + c(q,t) \tag{5.20}$$

and the potential energy $V = V(q,t)$ we can construct the Lagrangian

$$L = \frac{1}{2} a(q,t)\,\dot{q}^2 + b(q,t)\,\dot{q} + c(q,t) - V(q,t) \tag{5.21}$$

and the generalized momentum

$$p = \frac{\partial L}{\partial \dot{q}} = a(q,t)\,\dot{q} + b(q,t) \tag{5.22}$$

which makes the Hamiltonian

$$\begin{aligned} H = p\dot{q} - L &= a(q,t)\,\dot{q}^2 + \cancel{b(q,t)\,\dot{q}} - \frac{1}{2} a(q,t)\,\dot{q}^2 - \cancel{b(q,t)\,\dot{q}} - c(q,t) + V(q,t) \\ &= \frac{1}{2} a(q,t)\,\dot{q}^2 - c(q,t) + V(q,t) \end{aligned} \tag{5.23}$$

The first two terms are only equal to T if $b = 0$ and $c = 0$, i.e., if $T = \frac{1}{2} a(q,t)\,\dot{q}^2$.

In the general case, $H = T + V$ if and only if $T = \sum_{k=1}^{f} \sum_{k'=1}^{f} A_{kk'}(\{q_{k''}\}, t)\,\dot{q}_k\,\dot{q}_{k'}$.

5.3 Example: 1-D Harmonic Oscillator

So, for a one-dimensional harmonic oscillator

$$T = \frac{1}{2} m\dot{x}^2 = \frac{p^2}{2m} \tag{5.24a}$$

$$V = \frac{1}{2} kx^2 \tag{5.24b}$$

which meets the conditions (quadratic kinetic energy, velocity-independent potential energy) so the Hamiltonian is just the total energy:

$$H(x,p) = T + V = \frac{p^2}{2m} + \frac{1}{2} kx^2 \tag{5.25}$$

Hamilton's equations are

$$\frac{\partial H}{\partial x} = kx = -\dot{p} \tag{5.26a}$$

$$\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x} \tag{5.26b}$$

Now look at an example where T is *not* purely quadratic ...

5.4 Rotating Coördinates

As we've seen before,

$$L = \frac{m\dot{x}^{*2}}{2} + \frac{m\dot{y}^{*2}}{2} - m\omega\dot{x}^*y^* + m\omega\dot{y}^*x^* \underbrace{-V(x^*,y^*) + \frac{m\omega^2x^{*2}}{2} + \frac{m\omega^2y^{*2}}{2}}_{-V_{\text{eff}}(x^*,y^*)} \tag{5.27}$$

So the generalized momenta are

$$p_{x^*} = m\dot{x}^* - m\omega y^* \tag{5.28a}$$
$$p_{y^*} = m\dot{y}^* + m\omega x^* \tag{5.28b}$$

which can be inverted to give

$$\dot{x}^* = \frac{p_{x^*}}{m} + \omega y^* \tag{5.29a}$$
$$\dot{y}^* = \frac{p_{y^*}}{m} - \omega x^* \tag{5.29b}$$

We want to construct

$$H = p_{x^*}\dot{x}^* + p_{y^*}\dot{y}^* - L \tag{5.30}$$

The straightforward approach says to substitute for $\dot{x}^*$ and $\dot{y}^*$. But the algebra is easier if we substitute for p_{x^*} and p_{x^*} and then back again:

$$\begin{aligned} H &= m\dot{x}^{*2} - \cancel{m\omega\dot{x}^*y^*} + m\dot{y}^{*2} - \cancel{m\omega\dot{y}^*x^*} - \frac{m\dot{x}^{*2}}{2} - \frac{m\dot{y}^{*2}}{2} + \cancel{m\omega\dot{x}^*y^*} - \cancel{m\omega\dot{y}^*x^*} + V_{\text{eff}} \\ &= \frac{1}{2}m\left(\frac{p_{x^*}}{m} + \omega y^*\right)^2 + \frac{1}{2}m\left(\frac{p_{y^*}}{m} - \omega x^*\right)^2 + V(x^*,y^*) - \frac{m\omega^2x^{*2}}{2} - \frac{m\omega^2y^{*2}}{2} \\ &= \frac{p_{x^*}^2}{2m} + \frac{p_{y^*}^2}{2m} + \omega y^*p_{x^*} - \omega x^*p_{y^*} + V(x^*,y^*) \end{aligned} \tag{5.31}$$

From this Hamiltonian we get Hamilton's equations:

$$\frac{\partial H}{\partial p_{x^*}} = \frac{p_{x^*}}{m} + \omega y^* = \dot{x}^* \tag{5.32a}$$
$$\frac{\partial H}{\partial p_{y^*}} = \frac{p_{y^*}}{m} - \omega x^* = \dot{y}^* \tag{5.32b}$$
$$\frac{\partial H}{\partial x^*} = -\omega p_{y^*} + \frac{\partial V}{\partial x^*} = -\dot{p}_{x^*} \tag{5.32c}$$
$$\frac{\partial H}{\partial y^*} = \omega p_{x^*} + \frac{\partial V}{\partial y^*} = -\dot{p}_{y^*} \tag{5.32d}$$

(5.32e)

5.4.1 Time-Dependence of Hamiltonian

Note one consequence of Hamilton's equations:

$$\frac{dH}{dt} = \sum_{k=1}^{f} \left(\frac{\partial H}{\partial q_k} \dot{q}_k + \frac{\partial H}{\partial p_k} \dot{p}_k \right) + \frac{\partial H}{\partial t} = \sum_{k=1}^{f} (-\dot{p}_k \dot{q}_k + \dot{q}_k \dot{p}_k) + \frac{\partial H}{\partial t} \tag{5.33}$$

I.e.,

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \tag{5.34}$$

Because of Hamilton's equations, the implicit time dependence of the Hamiltonian "cancels out".

This tells us something we already know:

If H is a function of the $\{p_k\}$ and $\{q_k\}$ with no explicit time dependence, H is a constant of the motion.

We knew this because we saw last time

$$\left(\frac{\partial H}{\partial t} \right)_{\{q_k\},\{p_k\}} = - \left(\frac{\partial L}{\partial t} \right)_{\{q_k\},\{\dot{q}_k\}} \tag{5.35}$$

and we first saw the Hamiltonian as the thing which is conserved when $\frac{\partial L}{\partial t} = 0$.

6 Lightning Recap

1. For f *degrees of freedom* the Lagrangian depends on $\{q_k | k = 1 \dots f\}$ and $\{\dot{q}_k | k = 1 \dots f\}$ and maybe t. Physically

$$L(\{q_k\}, \{\dot{q}_k\}, t) = \underbrace{T}_{\text{kinetic}} - \underbrace{V}_{\text{potential; usually fcn only of } \{q_k\}} \tag{6.1}$$

   Mechanics given by Lagrange's equations:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial q_k} \qquad k = 1 \dots f \tag{6.2}$$

2. **Constraints and Lagrange Multipliers**

   There are two different was to handle constraints: Either you can choose coördinates such that the constraints are automatically satisfied *or* use Lagrange multipliers (which are useful for getting the constraining forces).

   In the Lagrange multiplier method, the Lagrangian

$$L(\{q_k | k = 1 \dots f + c\}, \{\dot{q}_k | k = 1 \dots f + c\}, t) = T - V \tag{6.3}$$

   needs to be modified to enforce the constraints

$$h_j(\{q_k\}) = 0 \qquad j = 1 \dots c \tag{6.4}$$

The modified Lagrangian is

$$\widetilde{L}(\{q_k\},\{\lambda_j\},\{\dot{q}_k\},t) = L(\{q_k\},\{\dot{q}_k\},t) + \sum_{j=1}^{c} \lambda_j h_j(\{q_k\}) \tag{6.5}$$

The equations of motion are then the Lagrange equations for $\widetilde{L}$:

$$\frac{d}{dt}\frac{\partial \widetilde{L}}{\partial \dot{q}_k} = \frac{\partial \widetilde{L}}{\partial q_k} \tag{6.6a}$$

$$0 = \frac{\partial \widetilde{L}}{\partial \lambda_j} \tag{6.6b}$$

This works physically because the (unknown) constraining forces are perpendicular to the surface of constraint.

3. **Conserved quantities**

$$\frac{\partial L}{\partial q_k} = 0 \text{ for some } k \qquad \Rightarrow \qquad p_k = \frac{\partial L}{\partial \dot{q}_k} \text{ conserved for that } k \tag{6.7a}$$

$$\frac{\partial L}{\partial t} = 0 \qquad \Rightarrow \qquad H = \sum_{k=1}^{f} p_k \dot{q}_k - L \text{ conserved} \tag{6.7b}$$

7 Review of Lagrangian Mechanics

1. Basic Formulation: f degrees of freedom, generalized coördinates $\{q_k | k = 1 \ldots f\}$, Lagrangian $L(\{q_k\},\{\dot{q}_k\},t)$

   Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial q_k} \tag{7.1}$$

   are equivalent to Newton's laws when $L = T - V$.

   (a) Derivation starting from Newton's laws in Cartesian coördintes $\{X_\ell | \ell = 1 \ldots 3N\}$ for N particles in 3 dimensions with

$$T(\{\dot{X}_\ell\}) = \frac{1}{2} M_\ell \dot{X}_\ell^2 \qquad \text{and} \qquad V(\{X_\ell\},t) \tag{7.2}$$

   Newton's laws are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{X}_\ell}\right) = \frac{d}{dt}(M_\ell \dot{X}_\ell) = -\frac{\partial V}{\partial X_\ell} \tag{7.3}$$

   (The one-particle version of this is $\frac{d}{dt}(m\dot{\vec{r}}) = \vec{\nabla} V$.)
   Converting derivatives using $X_\ell(\{q_k\},t)$ gives $T(\{q_k\},\{\dot{q}_k\},t)$ and $V(\{q_k\},t)$.

(b) Justification for $f < 3N$ comes from reduction of constraint problem.

(c) Can also start from Lagrangian, e.g., the electromagnetic Lagrangian

$$L = \frac{1}{2}m(\dot{\vec{r}} \cdot \dot{\vec{r}}) - Q\varphi(\vec{r}, t) + Q\dot{\vec{r}} \cdot \vec{A}(\vec{r}, t) \tag{7.4}$$

2. Constraints: Given $f + c$ coördinates but only f degrees of freedom, there are c constraints[4]

$$h_j(\{q_k\}, t) = 0 \qquad j = 1 \ldots c \tag{7.5}$$

Modify ordinary Lagrangian $L(\{q_k\}, \{\dot{q}_k\}, t)$ by adding terms involving new "coördinates" $\{\lambda_j | j = 1 \ldots c\}$

$$\widetilde{L}(\{q_k\}, \{\lambda_j\}, \{\dot{q}_k\}, \{\dot{\lambda}_j\}, t) = L(\{q_k\}, \{\dot{q}_k\}, t) + \sum_{j=1}^{c} \lambda_j h_j(\{q_k\}, t) \tag{7.6}$$

The equations of motion are then the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \dot{q}_k}\right) = \frac{\partial \widetilde{L}}{\partial q_k} = \frac{\partial L}{\partial q_k} + \sum_{j=1}^{c} \lambda_j \frac{\partial h_j}{\partial q_k} \tag{7.7a}$$

$$0 = \frac{d}{dt}\left(\frac{\partial \widetilde{L}}{\partial \dot{\lambda}_j}\right) = \frac{\partial L}{\partial \lambda_j} = h_j \tag{7.7b}$$

(7.7a) are the equations of motion including the constraining forces, and (7.7b) are the constraints.

(a) Method works because constrainting forces are perpendiculat for surface of constraint $\vec{F}_{\text{constraint}} \propto \vec{\nabla} h$; the λ_j are proportional to the constraining forces.

(b) **<u>WARNING</u>!** Do not impose constraints when constructing Lagrangian in Lagrange multiplier method. E.g., include $\frac{1}{2}m\dot{y}^2$ in Lagrangian even if $y = 0$ is a constraint.

(c) Constraints can also be handled by reduced Lagrangian with only f coördinates, but then you don't get the constraining forces.

3. Conservation Laws

(a) If Lagrangian is independent of a coördinate, the corresponding conjugate momentum is a constant of the motion.

$$\text{i.e., if } \frac{\partial L}{\partial q_k} = 0, \qquad \text{then } \frac{d}{dt}p_k = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0 \text{ as a result of Lagrange's eqns} \tag{7.8}$$

[4]Sometimes the constraints are written as $h_j(\{q_k\}, t) = a_j$ where a_j is some constant, but we can always write them in the form (7.5) by constructing $h^{\text{new}}(\{q_k\}, t) = h^{\text{old}}(\{q_k\}, t) - a_j = 0$.

(b) If the Lagrangian is independent of time (no explicit time dependence, i.e., $\left(\frac{\partial L}{\partial t}\right)_{\{q_k\},\{\dot{q}_k\}} = 0$, then

$$H = \sum_{k=1}^{f} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L \tag{7.9}$$

is a constant, i.e., $\frac{dH}{dt} = 0$ (this is the total derivative along the trajectory which satisfies Lagrange's equations).

H is "usually" the total energy. specifically, if

i. V depends only on $\{q_k\}$ and t, not $\{\dot{q}_k\}$

ii. T is purely quadratic in the $\{\dot{q}_k\}$, i.e.,

$$T = \sum_{k=1}^{f} \sum_{k'=1}^{f} \frac{1}{2} A_{kk'}(\{q_{k''}\}, t)\, \dot{q}_k\, \dot{q}_{k'} \tag{7.10}$$

(the most general form also has $\sum_{k=1}^{f} B_k(\{q_{k'}\}, t)\, \dot{q}_k$ and $T_0(\{q_k\}, t)$ terms)

then $\sum_{k=1}^{f} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} = 2T$ and $H = T + V = E$.

4. Hamiltonian Mechanics

Change of variables (Legendre transform) from $L(\{q_k\}, \{\dot{q}_k\}, t)$ to $H(\{q_k\}, \{p_k\}, t)$.

Hamilton's equations are

$$\dot{q}_k = \frac{dq_k}{dt} = \left(\frac{\partial H}{\partial p_k}\right) \tag{7.11a}$$

$$\dot{p}_k = \frac{dp_k}{dt} = -\left(\frac{\partial H}{\partial q_k}\right) \tag{7.11b}$$

(7.11a) is a derivative at constant $\{q_{k'}\}$, t, and $\{p_{k'} | k' \neq k\}$; (7.11b) is a derivative at constant $\{q_{k'} | k' \neq k\}$, t, and $\{p_{k'}\}$, *not* at constant $\{\dot{q}_{k'}\}$ because H is not written as a function of velocities.

There are $2k$ first-order equations which are equivalent to the k second-order Lagrange's equations.

(a) The Hamiltonian is defined by

$$H = \sum_{k=1}^{f} p_k \dot{q}_k - L \tag{7.12}$$

with a transformation of arguments via

$$p_k(\{q_{k'}\}, \{\dot{q}_{k'}\}, t) = \frac{\partial L}{\partial \dot{q}_k} \tag{7.13}$$

which can be inverted to get

$$\dot{q}_k(\{q_{k'}\}, \{p_{k'}\}, t) \tag{7.14}$$

(b) The method is most easily derived by implicit differentiation:

$$
\begin{aligned}
dH &= \sum_{k=1}^{f} \left(\frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial p_k} dp_k \right) + \frac{\partial H}{\partial t} dt \\
&= \sum_{k=1}^{f} \left(p_k \, d\dot{q}_k \dot{q}_k \, dp_k - \frac{\partial L}{\partial q_k} dq_k - \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k \right) - \frac{\partial L}{\partial t} dt \\
&= \sum_{k=1}^{f} \Bigg[\underbrace{-\frac{\partial L}{\partial q_k}}_{\substack{=-\dot{p}_k \text{ by} \\ \text{Lagrange} \\ \text{eqns}}} dq_k + \dot{q}_k \, dp_k + \overbrace{\left(p_k - \frac{\partial L}{\partial \dot{q}_k} \right)}^{0 \text{ by defn of } p_k} d\dot{q}_k \Bigg] - \frac{\partial L}{\partial t} dt
\end{aligned}
\tag{7.15}
$$

A Appendix: Correspondence to Class Lectures

Date	Sections	Pages	Topics
2006 May 26	1–1.1.4	3–8	Motivation and Formalism
2006 May 30	1.1.5–1.2	9–14	Derivation and Application
2006 June 1	Prelim One		