1 Decomposition of Angular Momentum

a) Substitute Symon’s (4.19), the definition of angular momentum of particle $k$ about a point $Q$, into Symon’s (4.23), the total angular momentum of a system about $Q$. Expand the cross product $(\vec{r}_k - \vec{r}_Q) \times (\dot{\vec{r}}_k - \dot{\vec{r}}_Q)$ appearing inside the sum in the resulting expression for $\vec{L}_Q$ and use the definitions of the total mass $M$, center of mass $\vec{R}$, and total momentum $\vec{P}$ of the system to simplify your expression for $\vec{L}_Q$ so that only one of the four terms still explicitly contains the sum over $k$, and the rest only contain $M$, $\vec{R}$, $\vec{P}$, $\vec{r}_Q$, and $\dot{\vec{r}}_Q$.

b) Simplify the result of part a) in the special case where the point $Q$ is the origin of coordinates ($\vec{r}_Q = \vec{0}$). Call this the total angular momentum $\vec{L}$.

c) Simplify the result of part a) in the special case where the point $Q$ is the center of mass ($\vec{r}_Q = \vec{R}$). Call this the angular momentum $\vec{L}_{\text{com}}$ relative to the center of mass.

d) Use the results of parts b) and c) to find an expression for the total angular momentum $\vec{L}$ in terms of $\vec{R}$, $\vec{P}$, and $\vec{L}_{\text{com}}$.

2 Internal and External Potential Gravitational Forces

The gravitational potential energy of two point masses $m_1$ and $m_2$ moving in the external gravitational field of a point mass $m_0$ fixed at the origin is

$$V(\vec{r}_1, \vec{r}_2) = -\frac{G m_0 m_1}{r_1} - \frac{G m_0 m_2}{r_2} - \frac{G m_1 m_2}{r}$$

(2.1)

where

$$r_1 = |\vec{r}_1| = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

(2.2a)

$$r_2 = |\vec{r}_2| = \sqrt{x_2^2 + y_2^2 + z_2^2}$$

(2.2b)

$$r = |\vec{r}_1 - \vec{r}_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

(2.2c)

a) Make a sketch of the locations of the three masses and the vectors $\vec{r}_1$, $\vec{r}_2$, and $\vec{r} = \vec{r}_1 - \vec{r}_2$. Indicate the distances $r_1$, $r_2$, $r$. 

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b) Calculate the gradients \( \vec{\nabla}_1 \vec{r}_1, \vec{\nabla}_1 \vec{r}_2, \vec{\nabla}_2 \vec{r}_1, \vec{\nabla}_2 \vec{r}_2, \) and \( \vec{\nabla}_2 \vec{r} \). Express your answers both in Cartesian coordinates and then in terms of the vectors \( \vec{r}_1, \vec{r}_2, \vec{r} \) and the magnitudes \( r_1, r_2, r \).

c) Find the internal forces \( \vec{F}_1^i = -\vec{\nabla}_1 V_i \) and \( \vec{F}_2^i = -\vec{\nabla}_2 V_i \) and verify that the strong form of Newton’s third law holds, i.e., that the vectors \( \vec{F}_1^i \) and \( \vec{F}_2^i \) are equal in magnitude, opposite in direction, and directed along the line connecting the locations of masses 1 and 2.

d) Find the external forces \( \vec{F}_1^e = -\vec{\nabla}_1 V_e \) and \( \vec{F}_2^e = -\vec{\nabla}_2 V_e \).

e) Using the formalism of the two-body problem, find an exact expression for \( M \ddot{\vec{R}} \) in terms of \( m_0, m_1, m_2, \vec{r}_1, \) and \( \vec{r}_2, \) and their magnitudes. (Here \( M = m_1 + m_2 \) is the total mass and \( \vec{R} = (m_1 \vec{r}_1 + m_2 \vec{r}_2)/M \) is the center of mass vector of the two freely-moving particles.)

f) Using the formalism of the two-body problem, find an exact expression for \( \vec{\mu} \ddot{\vec{r}} \), where \( \vec{\mu} = m_1 m_2 / M \) is the reduced mass of the two freely-moving particles. Note that Symon’s equation (4.96) does not hold, and you will need to retain a term

\[
\vec{F}_t(\vec{r}_1, \vec{r}_2) = \mu \left( \frac{\vec{F}_1^e}{m_1} - \frac{\vec{F}_2^e}{m_2} \right) \tag{2.3}
\]

Evaluate \( \vec{F}_t \) explicitly in terms of \( \vec{r}_1 \) and \( \vec{r}_2 \).

g) (bonus) In the limit that particles 1 and 2 are much closer to each other than they are to the origin, \( R = |\vec{R}| \gg r \), things simplify somewhat. Using Symon’s (4.92–4.93), expand your result for \( M \ddot{\vec{R}} \) to zeroth order and your result for \( \vec{F}_t \) to first order in the small parameter \( \xi = r/R \). Explicitly, this means

i) In your expression for \( M \ddot{\vec{R}} \), just replace \( \vec{r}_1 \) and \( \vec{r}_2 \) with \( \vec{R} \) (and similarly for their magnitudes) and you should get an answer which depends only on the properties of the 1-2 system as a whole (\( M \) and \( \vec{R} \)). What does this correspond to physically?

ii) In your expression for \( \vec{F}_t \), you’ll need to substitute in Symon’s (4.92–4.93) with \( \vec{r} = \xi \vec{R} \) and \( r = \xi R \) into the results of part f) and then Taylor expand the result and keep the terms linear in \( \xi \). Then you should be able to replace \( \xi \) with \( r/R \) and end up with an expression which depends on \( M, \mu, \vec{r}, \) and \( \vec{R} \). This describes the effects of the tidal field of the mass \( m_0 \) on the two-body system of masses 1 and 2.

### 3 Properties of a Mass Distribution

Consider the right triangular pyramid defined by

\[
\begin{align*}
x &\geq 0 \\
y &\geq 0 \\
z &\geq 0 \\
\frac{x}{a} + \frac{y}{b} + \frac{z}{c} &\leq 1
\end{align*}
\tag{3.1}
\]

with a constant density \( \rho \).
a) Sketch a (two-dimensional) constant-z cross-section of the pyramid. Indicate the $x$ and $y$ axes and label the coördinates of any points on the edges of the pyramid. (Some of these will depend upon $z$.)

b) Calculate the total mass $M$ of this pyramid by evaluating a triple integral over the volume of the pyramid. (Do not just use a formula for the volume of a pyramid.)

c) By performing triple integrals, calculate $X$, $Y$, and $Z$, the coördinates of the center of mass of the pyramid.