1 Partial Derivatives of the Hamiltonian

Note: when taking partial derivatives of the Hamiltonian, we usually consider it to be a function of coordinates and momenta rather than of velocities. In this problem, we will explicitly consider \( H \) as a function of different sets of arguments and compare the partial derivatives with different quantities held constant.

Consider the Lagrangian
\[
L(q, \dot{q}, t) = \frac{aq^2}{2} + b\dot{q}\sin\omega t - \frac{kq^2}{2}
\]
where \( a, b, \omega \) and \( k \) are all constants included to get the dimensions right.

a) Take the partial derivatives \( \left( \frac{\partial L}{\partial q} \right)_{\dot{q}, t} \), \( \left( \frac{\partial L}{\partial \dot{q}} \right)_{q, t} \), and \( \left( \frac{\partial L}{\partial t} \right)_{q, \dot{q}} \).

b) Find the conjugate momentum \( p(q, \dot{q}, t) = \left( \frac{\partial L}{\partial \dot{q}} \right)_{q, t} \).

c) Invert the results of part b) to obtain \( \dot{q}(q, p, t) \).

d) Construct the Hamiltonian \( H(q, \dot{q}, t) = p(q, \dot{q}, t) \dot{q} - L(q, \dot{q}, t) \), writing it first as a function of the coordinate and velocity with no reference to the momentum. (This is not how we usually do it, but we’re trying to prove a point here.)

e) Take the partial derivatives \( \left( \frac{\partial H}{\partial q} \right)_{p, t} \), \( \left( \frac{\partial H}{\partial p} \right)_{q, t} \), and \( \left( \frac{\partial H}{\partial t} \right)_{q, p} \). Show that \( \left( \frac{\partial H}{\partial q} \right)_{\dot{q}, t} \neq -\left( \frac{\partial L}{\partial q} \right)_{\dot{q}, t} \) and \( \left( \frac{\partial H}{\partial t} \right)_{q, \dot{q}} = -\left( \frac{\partial L}{\partial t} \right)_{q, \dot{q}} \).

f) Use the results of parts c) and d) to rewrite the Hamiltonian as a function \( H(q, p, t) \) of the coordinate and momentum with no reference to the velocity.

g) Take the partial derivatives \( \left( \frac{\partial H}{\partial q} \right)_{p, t} \), \( \left( \frac{\partial H}{\partial p} \right)_{q, t} \), and \( \left( \frac{\partial H}{\partial t} \right)_{q, p} \). These will be functions of \( q, p, \) and \( t \).

h) Use the results of part b) to write all three partial derivatives from part g) as functions of \( q, \dot{q}, \) and \( t \), and show that \( \left( \frac{\partial H}{\partial q} \right)_{p, t} = -\left( \frac{\partial L}{\partial q} \right)_{\dot{q}, t} \) and \( \left( \frac{\partial H}{\partial t} \right)_{q, p} = -\left( \frac{\partial L}{\partial t} \right)_{q, \dot{q}} \).
2 Two-Body Problem Revisited

Consider the Lagrangian
\[ L = \frac{M \dot{X}^2}{2} + \frac{M \dot{Y}^2}{2} + \frac{M \dot{Z}^2}{2} + \frac{\mu r^2 \dot{\theta}^2}{2} + \frac{\mu r^2 \sin^2 \theta \dot{\phi}^2}{2} + \frac{GM\mu}{r} - MgZ \]
which you found in problem 2 on problem set 5.

a) Construct the six conjugate momenta \( p_X, p_Y, p_Z, p_r, p_\theta, \) and \( p_\phi \) as functions of the coördinates \( \{X, Y, Z, r, \theta, \phi\} \) and velocities \( \{\dot{X}, \dot{Y}, \dot{Z}, \dot{r}, \dot{\theta}, \dot{\phi}\} \).

b) Invert those relationships to find the six generalized velocities \( \dot{X}, \dot{Y}, \dot{Z}, \dot{r}, \dot{\theta}, \) and \( \dot{\phi} \) in terms of the coördinates and momenta.

c) Construct the Hamiltonian as a function of the coördinates and momenta with no reference to any of the velocities in your final result.

d) Write all twelve of Hamilton’s equations. Which coördinates are ignorable?

3 Principle of Least Action

Consider a family of curves \( x_\alpha(t) = x(t) + \alpha \xi(t) \), where \( \xi(t) \) is an otherwise arbitrary function which vanishes at times \( t_i \) and \( t_f \) [i.e., \( \xi(t_i) = 0 = \xi(t_f) \)].

a) Calculate the derivatives \( \frac{\partial x_\alpha}{\partial \alpha} \) and \( \frac{\partial \dot{x}_\alpha}{\partial \alpha} \) where \( \dot{x}_\alpha \) is the time derivative of \( x_\alpha(t) \) (implicitly at constant \( \alpha \), since \( \alpha \) is a single number and not a function of time).

b) Consider a function \( L(x, \dot{x}, t) \), from which we can derive a function \( L_\alpha(t) = L(x_\alpha(t), \dot{x}_\alpha(t), t) \). Use the chain rule to write \( \frac{\partial L_\alpha}{\partial \alpha} \) in terms of the partial derivatives \( \frac{\partial L}{\partial x} \bigg|_{x=x_\alpha} \) and \( \frac{\partial L}{\partial \dot{x}} \bigg|_{x=x_\alpha} \).

c) Define the function
\[ S(\alpha) = \int_{t_i}^{t_f} L_\alpha(t) \, dt \]
and use the results of the previous two parts to write \( S'(\alpha) \) as an integral containing \( \xi, \dot{\xi}, \frac{\partial L}{\partial x} \bigg|_{x=x_\alpha} \) and \( \frac{\partial L}{\partial \dot{x}} \bigg|_{x=x_\alpha} \).

d) Use integration by parts (i.e., \( \int_{t_i}^{t_f} x \frac{dy}{dx} \, dt = xy\bigg|_{t_i}^{t_f} - \int_{t_i}^{t_f} y \frac{dx}{dy} \, dt \)) to convert the term involving \( \dot{\xi} \) into a term involving \( \xi \).

e) Show that if \( x(t) \) satisfies the Lagrange equation \( \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \), then \( S(\alpha) \) has a local extremum at \( \alpha = 0 \).

\( S \) is called the action, and Lagrange’s equations are equivalent to the condition that the action be smaller for the classical trajectory than for any “nearby” trajectory.