Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems.

1 Periodically Accelerating Reference Frame

Symon Chapter Seven, Problem Two.

2 Rotational Oblateness of the Earth

Consider the Earth, rotating at a fixed angular velocity \( \vec{\omega} \). Define two sets of Cartesian coördinates, \((x, y, z)\) and \((x^*, y^*, z^*)\), each with its origin at the center of the Earth and its positive \( z \) (or \( z^* \)) axis pointed towards the North Pole along the Earth’s rotation axis, so that \( \vec{\omega} = \omega \hat{z} = \omega \hat{z}^* \). Let the unstarrred coördinate axes be fixed in space while the starred ones rotate along with the Earth.

a) Define spherical coördinates, \((r, \theta, \phi)\) and \((r^*, \theta^*, \phi^*)\), respectively, corresponding to the non-rotating and rotating Cartesian coördinate systems. Explain (or show) why \( r^* = r \) and \( \theta^* = \theta \), and also \( \hat{r}^* = \hat{r} \) and \( \hat{\theta}^* = \hat{\theta} \).

b) Assume that most of the matter in the Earth is spherically symmetric about its center, so that the gravitational field is well approximated by

\[
\vec{g} = -g(r) \hat{r} \tag{2.1}
\]

where the magnitude \( g(r) \) is a function of \( r \) alone. Find the effective gravitational field \( \vec{g}^{\text{eff}} \) including centrifugal effects in the co-rotating coördinate system, in terms of \( g(r) \), \( \omega \), \( r \), \( \theta \), and the basis vectors \( \hat{r} \) and \( \hat{\theta} \).

c) In the limit \( \omega \to 0 \), the Earth is just a sphere of radius \( R \) and mass \( M \), and the gravitational acceleration at its surface is

\[
g_0 = \frac{GM}{R^2}. \tag{2.2}
\]

Construct the dimensionless combination of \( \omega, g_0, \) and \( R \) which is proportional to \( \omega^2 \), and call this \( \varepsilon \).
d) Let the actual shape of the surface of the rotating Earth be given by

\[ r = R + \delta R(\theta) \]  

(2.3)

where \( \delta R(\theta) \) is a small correction of order \( \varepsilon \). If \( \vec{n} \) is a vector normal (perpendicular) to this surface, what is the ratio of components \( n_\theta/n_r \)?

e) Using the results of part b), construct the ratio \( g^\text{eff}_\theta/g^\text{eff}_r \). This should be proportional to \( \omega^2 \) and therefore also of order \( \varepsilon \).

f) In your answer to part e), set \( r \) to \( R \) and neglect the \( \omega^2 \) term appearing in the denominator. (These approximations are justified because anything more accurate would just add a correction of order \( \varepsilon^2 \).) Express this approximate ratio in terms of \( g_0 \), \( R \), \( \omega \), and \( \theta \). Verify that you get sensible results at the North Pole (\( \theta = 0 \)), the Equator (\( \theta = \pi/2 \)) and the South Pole (\( \theta = \pi \)).

g) By requiring \( g^\text{eff} \) to be normal to the surface of the Earth, obtain and solve a differential equation for \( \delta R(\theta) \). (Don’t forget to include the integration constant in the solution.) What is the difference between the polar and equatorial radii, in terms of \( g_0, R, \) and \( \omega \)?

h) Using the actual values for the Earth (\( g_0 = 9.8 \text{ m/s}^2 \), \( \omega = 2\pi/(24 \text{ hr}) \), and \( R = 6.4 \times 10^3 \text{ km} \)), evaluate the following ratios to two significant figures:

i) \( \varepsilon \), the “small” parameter defined in part c);
ii) The fractional decrease of the magnitude of \( g^\text{eff} \) at the equator relative to the poles;
iii) \( \frac{\delta R(\pi/2) - \delta R(0)}{R} \), the size of the equatorial bulge as a fraction of the Earth’s radius.

3 Deflection of a Falling Object Due to Coriolis Force

Do Symon Chapter Seven, Problem Seven. Also evaluate your expression for the displacement, for an object dropped from a height of 20 meters at a latitude of 30°N, and specify the direction of the deflection.