1 Central Force with Quadratic Potential

Consider a potential \( V(r) = \frac{1}{2}kr^2 \).

a) For a particle of mass \( m \) moving in this potential, with angular momentum \( L \), construct the effective potential \( V_{\text{eff}}(r) \) and sketch a plot of \( V_{\text{eff}}(r) \) versus \( r \).

b) For what values of total energy are there two turning points \( r_{\min} \) and \( r_{\max} \)? Find \( r_{\min} \) and \( r_{\max} \) in terms of the energy \( E \).

c) Use the function \( V_{\text{eff}}(r) \) to find the radius \( r_{\text{circ}} \) of a circular orbit with angular momentum \( L \). What is the total energy \( E_{\text{circ}} \) of this orbit?

d) For an energy only slightly larger than \( E_{\text{circ}} \), calculate the frequency \( \omega_R \) of the small radial oscillations about \( r_{\text{circ}} \). Calculate the angular frequency \( \omega_\Phi \) of the angular oscillations when \( r \approx r_{\text{circ}} \) and compare the two frequencies quantitatively. (Both frequencies should be expressed in terms of the parameters \( k, m, \) and \( L \), and not in terms of e.g., \( r_{\text{circ}} \) or \( E_{\text{circ}} \).)

2 Conic Sections

Demonstrate that the orbit
\[
 r(1 + \varepsilon \cos \phi) = \alpha
\]
with constants \( \alpha > 0 \) and \( \varepsilon \geq 0 \) is indeed a conic section with eccentricity \( \varepsilon \), semimajor axis \( \alpha/(1 - \varepsilon^2) \), and one focus at \( r = 0 \) as follows:

a) Consider the points \( \mathcal{P} \equiv (x, y), \mathcal{O} \equiv (0,0), \mathcal{F}_\pm \equiv (\pm 2c, 0) \), (where \( c > 0 \)) and the line \( \mathcal{L} \equiv x = 2p > 0 \). Calculate the following distances in Cartesian coordinates, then convert your results into the standard polar coordinates using \( x = r \cos \phi \) and \( y = r \sin \phi \), simplifying as much as possible.

i) the length \( d_{\mathcal{OP}} \) of the straight line segment from \( \mathcal{O} \) to \( \mathcal{P} \)

ii) the length \( d_{\mathcal{F}_\pm \mathcal{P}} \) of the straight line segment from \( \mathcal{F}_\pm \) to \( \mathcal{P} \)

iii) the distance \( d_{\mathcal{LP}} \) between the point \( \mathcal{P} \) and the line \( \mathcal{L} \)

b) A circle of radius \( a \) centered at \( \mathcal{O} \) is the set of all points a distance \( a \) from \( \mathcal{O} \):
\[
 d_{\mathcal{OP}} = a
\]

Show that when \( \varepsilon = 0 \), (2.1) is equivalent to (2.2) for a suitable choice of \( a \), and find this \( a \) in terms of \( \alpha \).
c) An ellipse of semimajor axis $a > 0$ with foci at $F_-$ and $O$ is the set of all points such that the sum of their distances from the two foci is $2a$:

$$d_{F_-P} + d_{OP} = 2a \quad (2.3)$$

Show that when $0 < \varepsilon < 1$, (2.1) is equivalent to (2.3) for a suitable choice of $a$ and $c$, and find these values in terms of $\alpha$ and $\varepsilon$. (Hint: this is easiest if you solve (2.3) for $d_{F_-P}$, square it, and set it equal to the square of the result from part a), using (2.1) to eliminate $\cos \phi$, and requiring equality for any value of $r$.)

d) A parabola with focus $O$ and directrix $L$ is the set of all points equidistant from $O$ and $L$:

$$d_{LP} = d_{OP} \quad (2.4)$$

Show that when $\varepsilon = 1$, (2.1) is equivalent to (2.4) for a suitable choice of $p$, and find this $p$ in terms of $\alpha$.

e) The left branch of a hyperbola of semimajor axis $a < 0$ with foci at $O$ and $F_+$ is the set of all points such that the difference of their distances from the two foci is $-2a > 0$:

$$d_{F_+P} - d_{OP} = -2a \quad (2.5)$$

Show that when $\varepsilon > 1$, (2.1) is equivalent to (2.5) for a suitable choice of $a$ and $c$, and find these values in terms of $\alpha$ and $\varepsilon$. (Hint: this is easiest if you solve (2.5) for $d_{F_+P}$, square it, and set it equal to the square of the result from part a), using (2.1) to eliminate $\cos \phi$, and requiring equality for any value of $r$.)

3 Circular Orbits in a Gravitational Field

Note: None of your answers to this problem should involve the constant $K$; you should use the relationship $K = -GMm$ to express them in terms of the masses of the attracting body and the test particle.

Consider a test particle of mass $m$ moving in a circular orbit of radius $R$ under the gravitational attraction of a body of mass $M$ fixed at the center of the circle.

a) Use Kepler’s third law to calculate the orbital speed $v$ as a function of $R$.

b) Express the total energy $E$ and angular momentum $L$ as functions of the radius $R$ of the orbit (and not of each other or $v$).

c) Use the result of part a) to find the kinetic energy $K$ as a function of $R$.

d) Write the potential energy $V(R)$ and verify that $T + V = E$.

e) Suppose we reduce the orbital energy from a satellite in such a way that it changes from one circular orbit to another. Do the following quantities increase or decrease?

i) orbital radius; ii) orbital speed; iii) orbital period

iv) kinetic energy; v) potential energy; vi) orbital angular momentum