1 Projectile Motion with Air Resistance (conclusion)

Recall that in problem 3 on Problem Set 6, you found that $y(t) = 0$ and that $x(t)$ and $z(t)$ obeyed the equations of motion

$$\ddot{x}(t) = -\frac{b}{m} \dot{x}(t) \quad (1.1a)$$

$$\ddot{z}(t) = -g - \frac{b}{m} \dot{z}(t) \quad (1.1b)$$

with initial conditions

$$x(0) = 0 \quad (1.2a)$$

$$z(0) = 0 \quad (1.2b)$$

and

$$\dot{x}(0) = v_0 \cos \alpha \quad (1.3a)$$

$$\dot{z}(0) = v_0 \sin \alpha \quad (1.3b)$$

a) Solve the equations of motion (1.1) with initial conditions (1.3) to find $\dot{x}(t)$ and $\dot{z}(t)$ in terms of the time $t$, the coordinates $x(t)$ and $z(t)$, and the parameters of the problem.

b) Solve the equations you got in part a) subject to the initial conditions (1.2) to find $x(t)$ and $z(t)$ and write the solution $\vec{r}(t)$ in terms of the time, the parameters of the problem, and the basis vectors $\hat{x}$, $\hat{y}$, $\hat{z}$.

c) If $T$ is the time when the projectile lands on level ground, write the equation which implicitly defines $T$ in terms of the parameters of the problem.

d) The result of part c) is called a transcendental equation because $T$ appears both within and outside a transcendental function. It cannot be solved exactly, but when the air resistance is small, we can find an approximate solution perturbatively. As a first step in this process, define the dimensionless quantities $\beta = \frac{b v_0}{mg}$ and $\mathcal{T} = \frac{g T}{v_0}$ and use these definitions to rewrite your implicit equation for $T$ as an equation relating only the dimensionless quantities $\mathcal{T}$, $\beta$, and $\alpha$. 

Show your work on all problems!
2 The Curl

a) If \( a(\vec{r}) \) is a scalar field and \( \vec{B}(\vec{r}) \) is a vector field, show, by explicit evaluation of the left- and right-hand sides in Cartesian coordinates, that

\[
\vec{\nabla} \times (a \vec{B}) = (\vec{\nabla} a) \times \vec{B} + a(\vec{\nabla} \times \vec{B}) .
\]  

(2.1)

b) Writing the “del operator” in spherical coordinates according to Symon’s equation (3.124) allows us to write the curl of a vector as

\[
\vec{\nabla} \times \vec{A} = \hat{r} \times \frac{\partial A}{\partial r} + \frac{\hat{\theta}}{r} \times \frac{\partial A}{\partial \theta} + \frac{\hat{\phi}}{r \sin \theta} \times \frac{\partial A}{\partial \phi}.
\]  

(2.2)

Use this, along with Symon’s equation (3.99), to calculate i) \( \vec{\nabla} \times \hat{r} \); ii) \( \vec{\nabla} \times \hat{\theta} \); iii) \( \vec{\nabla} \times \hat{\phi} \).

c) Using the results of parts a) and b), and writing a vector field \( \vec{A}(\vec{r}) \) as

\[
\vec{A}(\vec{r}) = A_r(r, \theta, \phi) \hat{r} + A_\theta(r, \theta, \phi) \hat{\theta} + A_\phi(r, \theta, \phi) \hat{\phi}
\]  

(2.3)

show that the curl in spherical coordinates is

\[
\vec{\nabla} \times \vec{A} = \left( \frac{1}{r} \frac{\partial A_\phi}{\partial \theta} - \frac{1}{r \sin \theta} \frac{\partial A_\theta}{\partial \phi} + \frac{\cos \theta}{r \sin \theta} A_\phi \right) \hat{r} + \left( \frac{1}{r \sin \theta} \frac{\partial A_r}{\partial \theta} - \frac{\partial A_\theta}{\partial r} - \frac{1}{r} A_\phi \right) \hat{\theta}
\]

\[
+ \left( \frac{\partial A_\theta}{\partial r} - \frac{1}{r} \frac{\partial A_r}{\partial \theta} + \frac{1}{r} A_\theta \right) \hat{\phi}
\]  

(2.4)

3 Force, Potential and Torque

Consider the force field

\[
\vec{F}(\vec{r}) = V_0 \frac{x \hat{x} + y \hat{y}}{x^2 + y^2}
\]  

(3.1)

a) By explicitly calculating the (three-dimensional) curl \( \vec{\nabla} \times \vec{F} \), verify that this is a conservative force.

b) Invert Symon’s equation (3.89) to obtain expressions for \( \hat{x} \), \( \hat{y} \) and \( \hat{z} \) in terms of the cylindrical coördinates \( \rho \), \( \phi \) and \( z \) and the basis vectors \( \hat{\rho} \), \( \hat{\phi} \), and \( \hat{z} \). Simplify your answer as much as possible.

c) Use Symon’s equation (3.87) and the results of part b) to write \( \vec{F} \) above entirely in terms of the cylindrical coördinates \( \rho \), \( \phi \) and \( z \) and the basis vectors \( \hat{\rho} \), \( \hat{\phi} \), and \( \hat{z} \) (and the constant \( V_0 \)). Simplify your answer as much as possible.

d) Working in cylindrical coordinates, find the potential energy \( V(\rho, \phi, z) \) such that \( \vec{F} = -\vec{\nabla} V \). Include in your result an arbitrary constant (so that you capture the entire family of possible potentials) and indicate its units.

e) Calculate the vector torque \( \vec{N} \) due to this force (in either Cartesian or cylindrical coördinates), and verify that the torque about the \( z \) axis vanishes.