Physics A301: Classical Mechanics II

Problem Set 10

Assigned 2003 April 28
Due 2003 May 5

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems.

1 Moment of Inertia of an Ellipsoid

Consider an ellipsoid $\mathcal{E}$ defined by[1]

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \tag{1.1}$$

with constant density ρ. In this problem, you will work out the components $\{I_{ij}\}$ of the inertia tensor for this solid.

a) Calculate the mass M in terms of ρ and the semiaxes a, b, and c of the ellipsoid as follows:

  i) Write the mass of the ellipsoid as a triple integral over x, y, and z. Don't write the limits of integration for the three integrals explicitly, but leave it as an "integral over $\mathcal{E}$".

  ii) Change variables in this integral from x to $\xi = x/a$, from y to $\eta = y/b$, and from z to $\zeta = z/c$. Write out the condition on ξ, η and ζ which describes the region of integration. (This should be a single equation analogous to (1.1).)

  iii) Change variables again from ξ, η, and ζ to σ, ϑ, and φ defined so that

$$\begin{aligned}\xi &= \sigma \sin\vartheta \cos\varphi \\ \eta &= \sigma \sin\vartheta \sin\varphi \\ \zeta &= \sigma \cos\vartheta\end{aligned}$$

  (As a reminder, the implied volume element is $d\xi\, d\eta\, d\zeta = \sigma^2 \sin\vartheta\, d\sigma\, d\vartheta\, d\varphi$.) Now you should be able to set explicit limits of integration for your σ, ϑ and φ integrals.

  iv) Evaluate the triple integral to obtain M as a function of ρ and the dimensions (a, b, and c) of the ellipsoid.

b) Calculate the integral

$$\iiint\limits_{\mathcal{E}} z^2\, dx\, dy\, dz$$

by the same method used in part a) of this problem.

[1]The lengths a, b, and c are called the *semiaxes* of the ellipsoid.

c) Find the values of the integrals

$$\iiint\limits_{\mathcal{E}} x^2\,dx\,dy\,dz$$

and

$$\iiint\limits_{\mathcal{E}} y^2\,dx\,dy\,dz$$

either explicitly or by analogy to the result of part b). (If you use the latter approach, be sure to explain your reasoning fully.)

d) Explain why the "mixed" integrals $\iiint\limits_{\mathcal{E}} xy\,dx\,dy\,dz$, $\iiint\limits_{\mathcal{E}} yz\,dx\,dy\,dz$, and $\iiint\limits_{\mathcal{E}} xz\,dx\,dy\,dz$ all vanish.

e) Use the results of this problem to find the components

$$\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

of the inertia tensor for $\mathcal{E}$ about its center of mass in terms of M and the dimensions of the ellipsoid. (Note that this will mean substituting in for ρ using the results of part a).)

2 Principal Axes of Inertia

Consider a rigid body which consists of two point masses, each of mass $M/2$, separated by a massless rigid rod of length $2a$.

a) Let the body coördinates (x, y, z) be chosen so that the masses are located at $\vec{x}_A = (0, a/2, a\sqrt{3}/2)$ and $\vec{x}_B = (0, -a/2, -a\sqrt{3}/2)$. Work out (by direct calculation) the components $\{I_{ij}\}$ of the inertia tensor and write them as a matrix

$$\mathbf{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \tag{2.1}$$

b) Define an alternate set of coördinates $(\overline{x}, \overline{y}, \overline{z})$ according to

$$\begin{pmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{pmatrix}}_{\mathcal{R}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{2.2}$$

i) What are the coördinates $(\overline{x}_A, \overline{y}_A, \overline{z}_A)$ and $(\overline{x}_B, \overline{y}_B, \overline{z}_B)$ of the two point masses?

ii) Using the results of part i), calculate directly the components $\{I_{\overline{k}\overline{\ell}}\}$ of the inertia tensor in this new basis and write them as a matrix

$$\overline{\mathbf{I}} = \begin{pmatrix} I_{\overline{x}\overline{x}} & I_{\overline{x}\overline{y}} & I_{\overline{x}\overline{z}} \\ I_{\overline{y}\overline{x}} & I_{\overline{y}\overline{y}} & I_{\overline{y}\overline{z}} \\ I_{\overline{z}\overline{x}} & I_{\overline{z}\overline{y}} & I_{\overline{z}\overline{z}} \end{pmatrix} \tag{2.3}$$

c) Calculate $\mathbf{R}\mathbf{I}\mathbf{R}^{\mathrm{T}}$ and verify it is equal to $\overline{\mathbf{I}}$.

3 Angular Momentum and Rotational Energy

Consider a rigid body with body axes $\vec{e}_x$, $\vec{e}_y$, and $\vec{e}_z$ chosen to lie along the principal axes of inertia so that the inertia tensor is diagonal.

a) Write the angular velocity components ω_x, ω_y, and ω_z in terms of the angular momentum components L_x, L_y, and L_z and the moments of inertia I_{xx}, I_{yy}, and I_{zz}.

b) Use the results of part a) to write the rotational kinetic energy (which we'll call E since it's the only form of energy we're worrying about in this problem) in terms of L_x, L_y, L_z, I_{xx}, I_{yy}, and I_{zz}, without reference to any of the angular velocity components.

c) Limit attention to the case where two of the moments of inertia are equal, $I_{xx} = I_{yy} = I_1 \neq I_{zz} = I_3$. Let α be the angle between the angular momentum vector $\vec{L}$ and the symmetry axis $\vec{e}_z$, so that $L_z = L\cos\alpha$ where $L = \sqrt{L_x^2 + L_y^2 + L_z^2}$ is the magnitude of the angular velocity vector. Use this to write E as a function of L and α (and the body parameters I_1 and I_3), eliminating all references to L_x, L_y, and L_z.

d) Calculate $\left(\frac{\partial E}{\partial \alpha}\right)_L$ and indicate for what values of $\alpha \in (0, \pi/2)$ it is positive, negative, or zero[2]

   i) for an oblate object ($I_3 > I_1$)

   ii) for a prolate object ($I_1 > I_3$)

e) Many interactions with the outside world and/or small corrections to the assumption of rigid body motion will reduce the rotational energy of an object while leaving its angular momentum unchanged. Assuming $0 < \alpha < \pi/2$, use your results from the previous part to determine whether such interactions will cause the angle between the symmetry axis and the angular momentum vector to increase or decrease

   i) for an oblate object ($I_3 > I_1$)

   ii) for a prolate object ($I_1 > I_3$)

[2] $\alpha \in (0, \pi/2)$ means α in the open interval defined by $0 < \alpha < \pi/2$