Physics A301: Classical Mechanics II

Problem Set 9

Assigned 2003 April 2
Due 2003 April 9

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems.

1 Infinitesimal Rotations

Recall from chapter one that the effects of a rotation can be written as

$$x'_i = \sum_{j=1}^{3} R_{ij} x_j$$

where $\{R_{ij}\}$ are the elements of some rotation matrix $\mathbf{R}$. (This can be written in matrix notation as $\mathbf{x}' = \mathbf{R}\mathbf{x}$, if we think of the position vector $\vec{x}$ as a column vector $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.) Rotations about the x, y, and z axes through angles θ_1, θ_2, and θ_3, respectively, can be written as

$$\mathbf{R}_1(\theta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix}; \quad \mathbf{R}_2(\theta_2) = \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}; \quad \mathbf{R}_3(\theta_3) = \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

a) Using the Taylor expansions for the sine and cosine, write the infinitesimal rotation matrices $\mathbf{R}_1(d\theta_1)$, $\mathbf{R}_2(d\theta_2)$, and $\mathbf{R}_3(d\theta_3)$, keeping terms to *second* order in the infinitesimal angles. Express your answers in the form

$$\mathbf{R}_1(d\theta_1) = \mathbf{A}_1^{(0)} + d\theta_1 \mathbf{A}_1^{(1)} + (d\theta_1)^2 \mathbf{A}_1^{(2)} + \mathcal{O}((d\theta_1)^3)$$

and likewise for $\mathbf{R}_2(d\theta_2)$ and $\mathbf{R}_3(d\theta_3)$, including the explicit forms for the matrices $\{\mathbf{A}_i^{(n)} | i = 1, \ldots, 3; n = 0, \ldots, 2\}$, which should contain nothing but pure numbers, and not depend on any angles.

b) By matrix multiplication, find $\mathbf{R}_1(d\theta_1)\mathbf{R}_3(d\theta_3) - \mathbf{R}_3(d\theta_3)\mathbf{R}_1(d\theta_1)$ including terms to second order in the infinitesimal angles (note that $(d\theta_1)^2$, $(d\theta_1)(d\theta_3)$, and $(d\theta_3)^2$ are all second-order terms), and show that it is in general non-zero to second order but zero to first order.

c) By working to *first* order in the infinitesimal angles, show that

   i) $\mathbf{R}_1(d\theta_1)\mathbf{R}_2(d\theta_2) - \mathbf{R}_2(d\theta_2)\mathbf{R}_1(d\theta_1) = \mathcal{O}((d\theta)^2)$

ii) $\mathbf{R}_2(d\theta_2)\mathbf{R}_3(d\theta_3) - \mathbf{R}_3(d\theta_3)\mathbf{R}_2(d\theta_2) = \mathcal{O}((d\theta)^2)$

where $\mathcal{O}((d\theta)^2)$ indicates any term which is second order in infinitesimal angles.

d) Using the results of this problem, work out to first order the explicit form of the rotation matrix

$$\mathbf{R}(\vec{d\theta}) = \mathbf{R}_1(d\theta_1)\mathbf{R}_2(d\theta_2)\mathbf{R}_3(d\theta_3)$$

corresponding to an infinitesimal rotation through $\vec{d\theta} = d\theta_1\vec{e}_1 + d\theta_2\vec{e}_2 + d\theta_3\vec{e}_3$

Note that the results of the previous two parts tell us that we'd get the same matrix no matter in what order we wrote the three individual axis rotations.

e) Show that this agrees with the result from class, i.e., that

$$\sum_{j=1}^{3} R_{ij}(\vec{d\theta})x_j = \vec{e}_i \cdot (\vec{d\theta} \times \vec{x}) = \sum_{k=1}^{3}\sum_{\ell=1}^{3} \epsilon_{ik\ell}\, d\theta_k x_\ell$$

2 Coriolis Force and Conservation of Angular Momentum

[Note: this problem is not an exercise in the technical formalism developed in this chapter; you're actually intended to build up each step by thinking about the physics and geometry of the situation and then in the end verify that your physically-derived results agree with the more general formal ones.]

Consider a flat turntable rotating counter-clockwise at a fixed angular velocity ω_z. Let a particle of mass m be instantaneously a distance r from the center, seen by an observer rotating with the turntable to be moving radially outward at a speed v_r.

a) What is the tangential component v'_ϕ of the particle's velocity as seen by an inertial observer not rotating with the turntable?

b) The angular momentum of the particle according to the inertial observer is $\ell'_z = mrv'_\phi$. (If this is not obvious, you should review the section on rotational motion in your Basic Physics text.) Write ℓ'_z in terms of the parameters of the problem (ω_z, m, r, and v_r).

c) After an infinitesimal time dt, the particle is a distance $r + dr$ from the center; find dr in terms of the parameters of the problem and dt. (Throughout this problem, you should drop any terms which are second order and higher in infinitesimal quantities.)

d) As a result of non-inertial effects, the particle may have picked up a small transverse velocity dv_ϕ as seen by the co-rotating observer. In terms of this unknown dv_ϕ, what is the transverse component of the particle's velocity as seen by the inertial observer? (This will be the transverse velocity of a fixed point on the turntable $r + dr$ from the center, plus dv_ϕ.)

e) Using the results of parts c) and d), work out the angular momentum $\ell'_z + d\ell'_z$ of the particle after an infinitesimal time dt, as seen by the inertial observer, in terms of the parameters of the problem, dt and the unknown dv_ϕ.

f) If the particle is moving freely and not subject to any torques, the angular momentum as measured by an inertial observer will be conserved, i.e., $\ell'_z + d\ell'_z = \ell'_z$. Use this condition to solve for dv_ϕ in terms of the parameters of the problem and dt.

g) Verify that this apparent angular acceleration seen by the rotating observer, needed to conserve angular momentum, has the same magnitude and direction as the Coriolis acceleration derived formally in class and in the text.

3 For the World is Hollow, And I Have Touched the Sky

Imagine living on the inside of a cylindrical shell of radius R and negligible mass, rotating at a constant angular speed ω. Define a right-handed coördinate system co-rotating with the cylinder, with the z axis pointed so that an outside observer sees the cylinder (and the x and y axes) rotating counter-clockwise around the z axis. (I.e., counter-clockwise as seen by someone who has the positive z axis pointed at them.) Express all your answers in this problem in terms of the co-rotating basis vectors $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$ pointed along these axes.

a) What is the angular velocity vector $\vec{\omega}$?

b) What is the position vector $\vec{x}_{\mathcal{P}}$ of the point $\mathcal{P}$ where the surface of the cylinder meets the x axis?

c) Calculate the centrifugal force experienced by a mass m at the point $\mathcal{P}$. Verify that this appears to be "down" from the point of view of someone standing on the inside of the cylinder.

d) Define cardinal directions (coined by Larry Niven in *Ringworld*) from the point of view of someone standing on the inside of the ring:

- *Up* is towards the center of the cylinder.
- *Down* is away from the center of the cylinder.
- *Spinward* is along the curve of the cylinder in the direction in which a non-rotating observer sees the surface moving.
- *Antispinward* is along the curve of the cylinder in the opposite direction.
- *Port* is parallel to the axis of the cylinder, to the left of someone standing up on the inside of the cylinder facing spinward.
- *Starboard* is parallel to the axis of the cylinder, to the right of someone standing up on the inside of the cylinder facing spinward.

Note that spinward, port, and up form a right-handed triple just as East, North and up do on Earth (If we tried to define North, South, East and West in this problem, we would get confused because we're used to standing on the outside of a sphere, not the inside of a cylinder.)

What is the velocity vector (in the rotating reference frame) of an object instantaneously at the point $\mathcal{P}$ moving at speed v if the direction of motion is

i) up

ii) spinward

iii) port

e) What is the Coriolis force on each of the objects considered in the previous part of this problem. In addition to expressing your answers in terms of components in the basis $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$, note in which of the six cardinal directions defined above it points.

f) If R is the radius of the Earth, what angular speed is needed to make the centrifugal force found in part lead to an acceleration of $9.8\,\mathrm{m/s^2}$?

g) For the angular speed found in part f), calculate the magnitude of the acceleration associated with the Coriolis force found in part e) if the speed is $30\,\mathrm{m/s}$.