Physics A301: Classical Mechanics II

Problem Set 7

Assigned 2003 March 17
Due 2003 March 24

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems.

This problem set gives you an opportunity to explore some of the consequences of the orbit

$$\frac{\alpha}{r} = 1 + \varepsilon \cos\phi \tag{0.1}$$

where the orbital parameters are related to the constants of the motion (total energy E and angular momentum ℓ) by

$$\alpha = \frac{\ell^2}{GM\mu^2} \tag{0.2a}$$

$$\varepsilon = \sqrt{1 + \frac{2E\ell^2}{G^2M^2\mu^3}} \tag{0.2b}$$

(None of your answers on this problem set should involve the constant k; you should use the relationship $k = GM\mu$ to express them in terms of the total mass and reduced mass of the system.)

1 Circular Orbits ($\varepsilon = 0$)

Consider a circular orbit of radius α.

a) Use Kepler's third law to calculate the orbital speed v as a function of α.

b) Express the total energy E and angular momentum ℓ as functions of the radius α of the orbit (and not of each other or v).

c) Use the result of part a) to find the kinetic energy K as a function of α.

d) Write the potential energy $U(\alpha)$ and verify that $T + U = E$.

e) Suppose we reduce the orbital energy from a satellite in such a way that it changes from one circular orbit to another. Do the following quantities increase or decrease?

i) orbital radius; ii) orbital speed; iii) orbital period

iv) kinetic energy; v) potential energy; vi) orbital angular momentum

2 Elliptical Orbits ($0 < \varepsilon < 1$)

Note that most of the results in this problem can sanity-checked by comparing them to the results of problem 1 in the $\varepsilon \to 0$ limit.

a) For elliptical orbits it is convenient to work with a and ε rather than α and ε, where $a = \frac{\alpha}{(1-\varepsilon^2)}$ is the semimajor axis. Rewrite the orbit (0.1) in terms of a and ε, with no reference to α.

b) Find the total energy E and angular momentum ℓ as functions of the orbital parameters a and ε. Remember to write your answer in terms of G, M, and μ rather than k.

c) Since the distance of the orbiting body from the center of attraction is changing, the potential energy U is not a constant of the motion, but varies as the body's r and ϕ coördinates change. Write the potential energy U for an orbit with a given a and ε as a function of the azimuthal coördinate ϕ, with no explicit reference to r.

d) Find the kinetic energy $T = E - U$ as a function of ϕ and use this to calculate the orbital speed v (the magnitude of the instantaneous velocity vector, *not* any component of it in a standard coördinate system) for an orbit of a given a and ε as a function only of the angular coördinate ϕ.

e) Use your result from part d) to find the speeds v_P at periapse ($\phi = 0$) and v_A at apoapse ($\phi = \pi$).

f) Consider the situation illustrated in figure 8-10 of Marion & Thornton, where we have circular orbits of radii r_1 and $r_2 > r_1$, respectively, and an elliptical orbit of perihelion distance $r_{\text{min}} = r_1$ and aphelion distance $r_{\text{max}} = r_2$ which is just tangent to both of the circular orbits. Find the semimajor axis a and eccentricity ε of the elliptical orbit and use the results of this problem and the previous one to find expressions for v_1, v_2 (the orbital speed of the inner and outer circular orbits, respectively), v_P, and v_A (the instantaneous orbital speeds at perihelion and aphelion, respectively, of the elliptical orbit). Order these four speeds from least to greatest.

3 Hyperbolic Orbits ($\varepsilon > 1$)

Far away from the center of attraction, a hyperbolic orbit asymptotes to a straight line. The *impact parameter* s is defined as the distance from the origin of the closest point on that straight line. (Physically, the impact parameter is the distance by which the incoming body would miss the origin if there were no gravitational attraction.) In this problem, you will consider hyperbolic orbits parametrized not by α and ε, but by s and the speed v_∞ with which the body moves far from the origin.

a) Obtain the total energy E as a function of s and v_∞, by considering the asymptotic situation, where the body is far from the center of attraction and all the energy is kinetic.

b) By considering the asymptotic motion, write the angular momentum ℓ as a function of s and v_∞. (Hint: you don't have to use any trig functions, only the definition of angular momentum and a geometric construction.)

c) Use your results from parts a) and b) to express the orbital parameters α and ε in terms of s and v_∞.

d) What is the ratio r_{min}/s of the actual distance of closest approach to the impact parameter, as a function of s and v_∞?

e) Let η be the angle between the asymptotes, so that $\pi - \eta$ is the angle (called δ in Marion & Thornton) through which the incoming body is deflected as a result of the gravitational interaction. From the geometry of the problem, work out the relationship between η and the maximum azimuthal angle ϕ_{max}, and use this to find $\cos\frac{\eta}{2}$ in terms of the orbital parameters α and ε

f) Use the results of part c) to write $\cos\frac{\eta}{2}$ and $\sin\frac{\eta}{2}$ in terms of s and v_∞.

g) Find the magnitude of the delta-vee

$$|\delta\vec{v}| = |\vec{v}_f - \vec{v}_i|$$

associated with this encounter, where $\vec{v}_i$ and $\vec{v}_f$ are the initial and final velocity vectors, respectively, which describe the motion long before and long after the encounter. (Hint: keep in mind that the magnitude of both $\vec{v}_i$ and $\vec{v}_f$ is v_∞ and the angle between them is the deflection angle $\delta = \pi - \eta$.