Physics A301: Classical Mechanics II

Problem Set 6

Assigned 2003 March 7
Due 2003 March 14

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems.

1 Central Force with Quadratic Potential

Consider a potential $U(r) = \frac{1}{2}kr^2$.

a) For a particle of mass μ moving in this potential, with angular momentum ℓ, construct the effective potential $V(r)$ and sketch a plot of $V(r)$ versus r.

b) Use the function $V(r)$ to find the radius of a circular orbit with angular momentum ℓ. What is the total energy E of this orbit?

c) For what values of total energy are there two turning points r_{min} and r_{max}? Find r_{min} and r_{max} in terms of the energy E.

d) Calculate the angle

$$\Delta\phi = 2\int_{r_{\text{min}}}^{r_{\text{max}}} \frac{\dot{\phi}}{\dot{r}} dr$$

through which the particle moves during a complete radial cycle. Under what circumstances will this be a rational multiple of 2π?

2 Inverse-Cube Correction to Inverse Square Force

Consider the force $\vec{F} = \left(-\frac{k}{r^2} - \frac{\lambda}{r^3}\right)\vec{e}_r$ arising from a potential

$$U(r) = -\frac{k}{r} - \frac{\lambda}{2r^2}$$

a) Use the effective potential $V(r)$ to describe qualitatively the radial motion of a particle with total energy $E < 0$ and angluar momentum $\ell > 0$ moving in this potential. For what values of λ does the radial motion have two turning points?

b) Take λ to lie in the range for which the radial motion has two turning points.

  i) Calculate $\Delta\phi$ for a full cycle of the radial motion.

ii) Find the orbital equation relating r and ϕ. (Hint: the calculation is very similar to the one for a pure inverse-square law.) How does this result differ from the result for a pure inverse-square force?

3 Conic Sections

Demonstrate that the orbit

$$r(1+\varepsilon\cos\phi)=\alpha \tag{3.1}$$

with constants $\alpha>0$ and $\varepsilon\geq 0$ is indeed a conic section with eccentricity ε, semimajor axis $\alpha/(1-\varepsilon^2)$, and one focus at $r=0$ as follows:

a) Consider the points $\mathcal{P}\equiv(x,y)$, $\mathcal{O}\equiv(0,0)$, $\mathcal{F}_\pm\equiv(\pm 2c,0)$, (where $c>0$) and the line $\mathcal{L}\equiv x=2p>0$. Calculate the following distances in Cartesian coördinates, then convert your results into the standard polar coördinates using $x=r\cos\phi$ and $y=r\sin\phi$, simplifying as much as possible.

   i) the length $d_{\mathcal{OP}}$ of the straight line segment from $\mathcal{O}$ to $\mathcal{P}$

   ii) the length $d_{\mathcal{F}_\pm\mathcal{P}}$ of the straight line segment from $\mathcal{F}_\pm$ to $\mathcal{P}$

   iii) the distance $d_{\mathcal{LP}}$ between the point $\mathcal{P}$ and the line $\mathcal{L}$

b) A circle of radius a centered at $\mathcal{O}$ is the set of all points a distance a from $\mathcal{O}$:

$$d_{\mathcal{OP}}=a \tag{3.2}$$

Show that when $\varepsilon=0$, (3.1) is equivalent to (3.2) for a suitable choice of a, and find this a in terms of α.

c) An ellipse of semimajor axis $a>0$ with foci at $\mathcal{F}_-$ and $\mathcal{O}$ is the set of all points such that the sum of their distances from the two foci is $2a$:

$$d_{\mathcal{F}_-\mathcal{P}}+d_{\mathcal{OP}}=2a \tag{3.3}$$

Show that when $0<\varepsilon<1$, (3.1) is equivalent to (3.3) for a suitable choice of a and c, and find these values in terms of α and ε. (Hint: this is easiest if you solve (3.3) for $d_{\mathcal{F}_-\mathcal{P}}$, square it, and set it equal to the square of the result from part a), using (3.1) to eliminate $\cos\phi$, and requiring equality for any value of r.)

d) A parabola with focus $\mathcal{O}$ and directrix $\mathcal{L}$ is the set of all points equidistant from $\mathcal{O}$ and $\mathcal{L}$:

$$d_{\mathcal{LP}}=d_{\mathcal{OP}} \tag{3.4}$$

Show that when $\varepsilon=1$, (3.1) is equivalent to (3.2) for a suitable choice of p, and find this p in terms of α.

e) The left branch of a hyperbola of semimajor axis $a<0$ with foci at $\mathcal{O}$ and $\mathcal{F}_+$ is the set of all points such that the difference of their distances from the two foci is $-2a>0$:

$$d_{\mathcal{F}_+\mathcal{P}}-d_{\mathcal{OP}}=-2a \tag{3.5}$$

Show that when $\varepsilon>1$, (3.1) is equivalent to (3.5) for a suitable choice of a and c, and find these values in terms of α and ε. (Hint: this is easiest if you solve (3.5) for $d_{\mathcal{F}_+\mathcal{P}}$, square it, and set it equal to the square of the result from part a), using (3.1) to eliminate $\cos\phi$, and requiring equality for any value of r.)