Physics A301: Classical Mechanics II

Problem Set 3

Assigned 2003 January 31
Due 2003 February 7

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems.

1 Euler Equations with Higher Derivatives

Consider the functional

$$J[y] = \int_{x_i}^{x_f} \mathcal{J}(y(x), y'(x), y''(x), x)\, dx$$

By a derivation analogous to the one used to obtain the usual Euler Equation [e.g., equation (6.18) of Marion & Thornton], determine the equation satisfied by the path $y(x)$ which minimizes $J[y]$. (This equation will involve $\frac{\partial \mathcal{J}}{\partial y}$ and $\frac{\partial \mathcal{J}}{\partial y'}$ as well as $\frac{\partial \mathcal{J}}{\partial y''}$.) Be sure to show all your steps and explain your reasoning in your own words.

2 Hemispherical Bowl Revisited

Consider a mass m moving without friction inside a hemispherical bowl of radius R under the presence of a constant downward gravitational acceleration g. Let θ be the angle between the mass and the bottom of the bowl, and let ϕ be an azimuthal angle going around the bowl. (Curves of constant θ are circles lying in parallel horizontal planes, while curves of constant ϕ are semicircular cross-sections of the bowl.)

a) Construct the Lagrangian $L(\theta, \phi, \dot{\theta}, \dot{\phi})$.

b) Find the Euler-Lagrange equations, and write them as differential equations of the form

$$\ddot{\theta} = F_1(\theta, \phi, \dot{\theta}, \dot{\phi})$$
$$\ddot{\phi} = F_2(\theta, \phi, \dot{\theta}, \dot{\phi})$$

c) Let the mass be released from rest at a position $\phi = 0$, $\theta = \sin^{-1} \frac{A}{R}$; show that if $A \ll R$, θ obeys a simple harmonic oscillator equation of motion, and calculate the oscillation frequency.

3 Rolling Oscillations

Consider a cylindrical block which rolls without slipping in the bottom of a cylindrical tube. Let the block have length a, radius r, and mass m, and the tube have radius $R > r$. Let the angle θ measure how far up the side of the tube the cylinder is, and the angle ψ measure how much the cylinder has rotated.

a) Write the instantaneous translational velocity of the cylinder as a whole in terms of $\dot{\theta}$; use this to write the translational contribution to the kinetic energy.

b) Write the instantaneous angular velocity of the cylinder as a whole in terms of $\dot{\psi}$; use this, along with the fact that the moment of inertia of the cylinder is $\frac{1}{2}mr^2$ to write the rotational contribution to the kinetic energy.

c) Let the gravitational field be g, directed straight downwards. Explain why the difference in potential energy between two positions should be related to the difference in the height h_{COM} of the center of mass of the cylinder between the two positions, and write that expression for ΔU in terms of Δh_{COM}.

d) Define h_{COM} to be zero when the cylinder is at the bottom of the tube ($\theta = 0$), and find an expression for $h_{\mathrm{COM}}(\theta)$. (Note: be very careful in working out the geometry here. This is the trickiest part of the problem.)

e) Define the potential energy to be zero when $\theta = 0$, and find the potential energy as a function of θ.

f) Combine the kinetic and potential energies to obtain the Lagrangian $L(\theta, \psi, \dot{\theta}, \dot{\psi})$.

g) Because the cylinder is rolling without slipping, the angles θ and ψ are not independent. Work out the equation of constraint relating them as follows:

   i) Write the distance ℓ travelled along the surface of the tube in terms of the angle θ that the cylinder has moved up the tube.

   ii) This distance ℓ must also equal the length of the surface of the cylinder between the point that was in contact when it was at the bottom to the new point of contact. Use this to write ℓ in terms of the angle ψ through which the cylinder has rolled.

   iii) Set these two expressions for ℓ equal to each other to obtain the constraint. Express this in the form $g(\theta, \psi) = 0$.

h) Write the modified Lagrangian $\hat{L}(\theta, \psi, \dot{\theta}, \dot{\psi}, \lambda) = L(\theta, \psi, \dot{\theta}, \dot{\psi}) - \lambda g(\theta, \psi)$ and find the modified Euler-Lagrange equations, which will be three equations: a differential equation for $\ddot{\theta}$, a differential equation for $\ddot{\psi}$, and the constraint itself.

i) Eliminate λ from the first two equations to obtain a single differential equation containing both $\ddot{\theta}$ and $\ddot{\psi}$.

j) Use the constraint to replace ψ and its derivatives with θ and its derivatives and obtain a single equation of motion for θ. (This should not contain ϕ or λ.)