1 Shortest Distance in Three Dimensions

Consider paths connecting two points \((x_i, y_i, z_i)\) and \((x_f, y_f, z_f)\) in three-dimensional space. In particular, limit attention to paths which are single-valued in \(z\) so they can be described by functions \(x(z)\) and \(y(z)\).

a) What are \(x(z_i), y(z_i), x(z_f), y(z_f)\) (the boundary conditions on \(x(z)\) and \(y(z)\))? 

b) What is the infinitesimal distance \(d\ell\) between \((x_0, y_0, z_0)\) and \((x_0 + dx, y_0 + dy, z_0 + dz)\)?

c) If \((x_0, y_0, z_0)\) and \((x_0 + dx, y_0 + dy, z_0 + dz)\) both lie on the path described by functions \(x(z)\) and \(y(z)\), write \(dx\) and \(dy\) in terms of \(dz\) and those functions.

d) Write the infinitesimal distance \(d\ell\) between the points on the path for which \(z = z_0\) and \(z = z_0 + dz\).

e) What is the rate \(d\ell/dz\) at which the distance along the path given by \(x(z)\) and \(y(z)\) increases with \(z\)?

f) Write, as an integral over the coordinate \(z\) which parametrizes the path \((x(z), y(z), z)\), the total length \(L[x, y]\) of that path from \(z_i\) to \(z_f\). This should be expressed in terms of the functions \(x(z)\) and \(y(z)\) and their derivatives \(x'(z)\) and \(y'(z)\).

g) Using this result, identify the integrand \(L(x, y, x', y', z)\) in the functional

\[
L[x, y] = \int_{z_i}^{z_f} L(x(z), y(z), x'(z), y'(z), z) dz
\]

h) Calculate the partial derivatives \(\frac{\partial L}{\partial x}, \frac{\partial L}{\partial y}, \frac{\partial L}{\partial x'}, \text{ and } \frac{\partial L}{\partial y'}\).

i) Using Euler’s equations, find the differential equations satisfied by \(x(z)\) and \(y(z)\).

j) Find the particular solution to these differential equations subject to the boundary conditions in part a)
2 The Double Pendulum

Consider the system illustrated in the figure below: a mass \( m \) hangs from a fixed suspension point by a rod of length \( \ell \), and a second mass, also of mass \( m \), hangs from the first mass by another rod, also of length \( \ell \).

a) Define a coordinate system in which the origin is at the suspension point of the first pendulum, the \( x \) direction is to the right, and the \( y \) direction is up. Let the position the first mass be given by Cartesian coordinates \((x_1(t), y_1(t))\) and the second by Cartesian coordinates \((x_2(t), y_2(t))\). Write the kinetic energies \( T_1 \) and \( T_2 \) and potential energies \( U_1 \) and \( U_2 \) in terms of the Cartesian coordinates and velocities.

b) If the first rod makes an angle \( \theta_1(t) \) with the vertical and the second rod makes an angle \( \theta_2(t) \) with the vertical, write
   i) \( x_1(t) \) and \( y_1(t) \) in terms of \( \ell \) and \( \theta_1(t) \);
   ii) \( x_2(t) \) and \( y_2(t) \) in terms of \( \ell \), \( \theta_2(t) \), \( x_1(t) \), and \( y_1(t) \);
   iii) \( x_2(t) \) and \( y_2(t) \) in terms of \( \ell \), \( \theta_2(t) \), and \( \theta_1(t) \).
   iv) \( \dot{x}_1(t) \), \( \dot{y}_1(t) \), \( \dot{x}_2(t) \), and \( \dot{y}_2(t) \) in terms of \( \ell \), \( \theta_1(t) \), \( \theta_2(t) \), \( \dot{\theta}_1(t) \), and \( \dot{\theta}_2(t) \),

c) Write the kinetic energies \( T_1 \) and \( T_2 \) and potential energies \( U_1 \) and \( U_2 \) in terms of the parameters \( \ell \) and \( m \), the angles \( \theta_1(t) \) and \( \theta_2(t) \), and their derivatives \( \dot{\theta}_1(t) \) and \( \dot{\theta}_2(t) \).

d) Write the Lagrangian \( L(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, t) \) describing the system, with \( \theta_1 \) and \( \theta_2 \) taken as the generalized coordinates.

e) Use the Euler-Lagrange equations to find the equations of motion satisfied by \( \theta_1(t) \) and \( \theta_2(t) \).

You should not assume either angle is small at any point in this problem.

3 Accelerated Pendulum

Marion & Thornton Problem 7-2.