Physics A301: Classical Mechanics II

Problem Set 1

Assigned 2003 January 15
Due 2003 January 22

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems.

1 More Fermat's Principle

Use Fermat's Principle that light travels along the path which takes the minimum time, together with the fact that the speed of light in a medium with an index of refraction n is c/n, to determine the path $y(x)$ a light ray will take from an initial position $(x_i, y(x_i)) = (-a, a)$ to a final position $(x_f, y(x_f)) = (a, a)$ when the index of refraction is:

a) $n(x, y) = e^{y/a}$

b) $n(x, y) = \frac{y-b}{c}$

2 Geodesic on a Cylinder

Consider the path $z(\phi)$ connecting two points $(\phi_i, z(\phi_i) = z_i)$ and $(\phi_f, z(\phi_i) = z_f)$ on the surface of a right cylinder of radius R.

a) Construct an integral expression for the length of the path

b) Use the Euler equation to find the equation satisfied by the $z(\phi)$ which gives a local extremum of the length.

c) Solve this equation to obtain $z(\phi)$ in terms of ϕ_i, ϕ_f, z_i, and z_f.

d) Because of the periodicity of the cylindrical coördinate ϕ, each point on the surface of the cylinder can be described by infinitely many (z, ϕ) pairs. Explicitly, $(z, \phi + 2\pi n)$ is the same point, for any n. As a consequence, your answer for part c) actually refers to only one member of this family; replace it with a $z_n(\phi)$ which enumerates all the members of the family.

e) If $\phi_f = \phi_i$, calculate the length of a general member $z_n(\phi)$ of the family of locally minimum paths. Which value of n gives the absolute minimum?

3 Brachistochrone Through the Earth

Consider a tunnel drilled between two points on the surface of a spherical planet of uniform density, mass M, and radius R.

a) Inside the tunnel, at a distance r from the center of the planet, what is the magnitude and direction of the gravitational force on a particle of mass $m \ll M$?

b) What is the potential energy of such a particle, as a function of distance?

c) If this particle is dropped into the tunnel from the surface of the planet, assuming no friction, what is the speed with which it travels through the tunnel as a function of r?

d) Choose spherical coördinates so that both ends of the tunnel lie on the equator ($\theta = 0$) and assume that the entire tunnel lies in the equatorial plane ($\theta = 0$). If the path is written as $r(\phi)$, construct an integral expression for the time taken to traverse the entire tunnel, subject to the assumptions of part c).

e) Let the tunnel be constructed so that the transit time is a minumum. Find (but don't solve) the differential equation satisfied by $r(\phi)$.