Physics A300: Classical Mechanics I

Revised Problem Set 2

Assigned 2002 September 6
Due 2002 September 13

Show your work on all problems! Note that the answers to some problems are in the appendix of Marion & Thornton, but they should only be used to check your work, since the answers themselves are an insufficient solution to the problem.

1 An Identity Relating the Levi-Civita and Kronecker Delta Symbols (M & T 1-22)

a) Evaluate the sum \( \sum_{k=1}^{3} \varepsilon_{ijk} \varepsilon_{\ell mk} \) (which is actually 81 different sums, each containing three terms) by considering the result for all possible combinations of \( i, j, \ell, m \), that is:
   \begin{align*}
   \text{(a)} & \quad i = j \\
   \text{(b)} & \quad i = \ell \\
   \text{(c)} & \quad i = m \\
   \text{(d)} & \quad j = \ell \\
   \text{(e)} & \quad j = m \\
   \text{(f)} & \quad \ell = m \\
   \text{(g)} & \quad i \neq \ell \text{ or } m \\
   \text{(h)} & \quad j \neq \ell \text{ or } m
   \end{align*}

   Show that
   \[ \sum_{k=1}^{3} \varepsilon_{ijk} \varepsilon_{\ell mk} = \delta_{i\ell} \delta_{jm} - \delta_{im} \delta_{j\ell} \]

b) Use the result of part a) to prove
   \[ \vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} \]

2 Circular Motion

Consider a particle which moves at uniform speed in a circular trajectory of radius \( a \) at angular velocity \( \omega \) (so that it completes an orbit in a time \( 2\pi/\omega \)). Let the particle move counter-clockwise in the \( xy \)-plane so that it crosses the positive \( y \)-axis at \( t = 0 \).

a) Write the trajectory \( x(t), y(t) \) in Cartesian coordinates.

b) By taking time-derivatives, calculate the velocity \( \vec{v}(t) = \dot{r}(t) \) and acceleration \( \vec{a}(t) = \ddot{r}(t) \) in Cartesian coordinates.

c) Write the trajectory \( r(t), \phi(t) \) in plane polar coordinates.

d) By taking time-derivatives of those expressions, re-calculate the velocity and acceleration in polar coordinates. (You will need to use the time derivatives of the \( \vec{e}_r \) and \( \vec{e}_\phi \) unit vectors.)
3 Expressing an Unknown Vector in Term of its Known Cross and Dot Products with a Known Vector (M & T 1-13)

Suppose that all we know about a vector $\vec{X}$ is that
\[ \vec{A} \times \vec{X} = \vec{B} \]
and
\[ \vec{A} \cdot \vec{X} = \varphi \]
and that we know $\vec{A}$, $\vec{B}$, and $\varphi$. Find an expression for $\vec{X}$ in terms of the known quantities $\vec{A}$, $\vec{B}$, $\varphi$, and $|\vec{A}| = \sqrt{\vec{A} \cdot \vec{A}}$.

(Hint: Consider $\vec{A} \times \vec{B}$.)

4 Invariance of the Scalar Product

Show that the same expression holds for the dot product $\vec{A} \cdot \vec{B}$ in terms of the components of $\vec{A}$ and $\vec{B}$ in any orthonormal basis as follows:

a) Consider the matrix
\[ \Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} \end{pmatrix} \]
whose transpose is
\[ \Lambda^T = \begin{pmatrix} \Lambda_{11}^T & \Lambda_{12}^T & \Lambda_{13}^T \\ \Lambda_{21}^T & \Lambda_{22}^T & \Lambda_{23}^T \\ \Lambda_{31}^T & \Lambda_{32}^T & \Lambda_{33}^T \end{pmatrix} \]
Write an expression for the $(i,k)$th element of $\Lambda^T$ (i.e., $\Lambda_{ik}^T$) in terms of the elements of $\Lambda$.

b) Now let $\Lambda$ be orthogonal, so that
\[ \Lambda^T \Lambda = 1. \]
Summarize the nine components of this $3 \times 3$ matrix equation in a single equation with two free indices (i.e., write an equation relating the $(i,j)$th component of each side of the matrix equality.)

c) Use the result of part a) to simplify the result of part b).

d) Let the orthogonal matrix $\Lambda$ define a transformation between orthonormal bases so that the components of the vector $\vec{A}$ in the new basis are given in terms of the components in the old basis by
\[ A_k = \sum_{i=1}^{3} \Lambda_{ik} A_i \]
Use the result of part c) to show that
\[ \sum_{k=1}^{3} A_k B_k = \sum_{i=1}^{3} A_i B_i \]