Physics A300: Classical Mechanics I
Supplemental Exercises on Fourier Series
Fall 2002

1 Trigonometric Fourier Series

Consider a function $h(t)$ defined for $-\frac{T}{2} < t < \frac{T}{2}$. Assume it can be described by the expansion

$$h(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin \left(\frac{2\pi nt}{T}\right)$$

1. Write the complex conjugate $h^*(t)$.
   If $h(t)$ is a real function ($h^*(t) = h(t)$), what conditions must the coefficients $\{a_n|n = 0, \ldots \infty\}$ and $\{b_n|n = 1, \ldots \infty\}$ satisfy? What if it’s an imaginary function ($h^*(t) = -h(t)$)?

2. Write $h(-t)$, using the symmetry properties of the sine and cosine to express it as a Fourier series with different coefficients. If $h(t)$ is an even function ($h(-t) = h(t)$), which coefficients must vanish? Are there any restrictions on the others? What if it’s an odd function ($h(-t) = -h(t)$)?

3. Using the identities

$$\cos(\theta_1 + \theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2$$
$$\cos(\theta_1 - \theta_2) = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2$$
$$\sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2$$
$$\sin(\theta_1 - \theta_2) = \sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2$$

find expressions for

$$\cos \left(\frac{2\pi nt}{T}\right) \cos \left(\frac{2\pi mt}{T}\right)$$
$$\sin \left(\frac{2\pi nt}{T}\right) \sin \left(\frac{2\pi mt}{T}\right)$$
$$\cos \left(\frac{2\pi nt}{T}\right) \sin \left(\frac{2\pi mt}{T}\right)$$
$$\sin \left(\frac{2\pi nt}{T}\right) \cos \left(\frac{2\pi mt}{T}\right)$$

as sums and differences of trig functions.

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1If $z = x + iy$ is a complex number ($x$ and $y$ are both real numbers), the complex conjugate is defined as $z^* = x - iy$; a real number ($y = 0$ in this representation) is thus unchanged by complex conjugation (thus $x^* = x$); a pure imaginary number ($x = 0$) changes sign under complex conjugation (($iy)^* = -iy$).
4. Using these expressions, calculate the integrals

\[
\begin{align*}
\int_{-T/2}^{T/2} \cos \left( \frac{2\pi nt}{T} \right) \cos \left( \frac{2\pi mt}{T} \right) dt & \quad m \neq n \\
\int_{-T/2}^{T/2} \sin \left( \frac{2\pi nt}{T} \right) \sin \left( \frac{2\pi mt}{T} \right) dt & \quad m \neq n \\
\int_{-T/2}^{T/2} \cos \left( \frac{2\pi nt}{T} \right) \sin \left( \frac{2\pi mt}{T} \right) dt & \quad m \neq n \\
\int_{-T/2}^{T/2} \cos^2 \left( \frac{2\pi nt}{T} \right) dt \\
\int_{-T/2}^{T/2} \sin^2 \left( \frac{2\pi nt}{T} \right) dt \\
\int_{-T/2}^{T/2} \cos \left( \frac{2\pi nt}{T} \right) \sin \left( \frac{2\pi nt}{T} \right) dt
\end{align*}
\]

The results can be expressed more compactly using the “Kronecker delta”

\[
\delta_{mn} = \begin{cases} 
1 & m = n \\
0 & m \neq n 
\end{cases}
\]

Notice that setting \( m = 0 \) in the above expressions also gives expressions for

\[
\begin{align*}
\int_{-T/2}^{T/2} \cos \left( \frac{2\pi nt}{T} \right) dt \\
\int_{-T/2}^{T/2} \sin \left( \frac{2\pi nt}{T} \right) dt
\end{align*}
\]

5. Use the previous results to calculate the integrals

\[
\begin{align*}
\int_{-T/2}^{T/2} h(t) \cos \left( \frac{2\pi nt}{T} \right) dt \\
\int_{-T/2}^{T/2} h(t) \sin \left( \frac{2\pi nt}{T} \right) dt \\
\int_{-T/2}^{T/2} h(t) dt
\end{align*}
\]

and thus obtain expressions for \( \{a_n|n = 0, \ldots \infty\} \) and \( \{b_n|n = 1, \ldots \infty\} \).
1.1 Answers

1. \[ h^*(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{2\pi nt}{T} \right) + \sum_{n=1}^{\infty} b_n \sin \left( \frac{2\pi nt}{T} \right) \]

If \( h(t) \) is real, all the \( \{a_n\} \) and \( \{b_n\} \) are real. If \( h(t) \) is imaginary, all the \( \{a_n\} \) and \( \{b_n\} \) are imaginary.

2. \[ h(-t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{2\pi nt}{T} \right) - \sum_{n=1}^{\infty} b_n \sin \left( \frac{2\pi nt}{T} \right) \]

If \( h(t) \) is even, all the \( b_n \) vanish, and the \( a_n \) are unrestricted. If \( h(t) \) is odd, all the \( a_n \) vanish, and the \( b_n \) are unrestricted.

3. \[
\begin{align*}
\cos \left( \frac{2\pi mt}{T} \right) \cos \left( \frac{2\pi nt}{T} \right) &= \frac{1}{2} \left[ \cos \left( \frac{2\pi (m-n)t}{T} \right) + \cos \left( \frac{2\pi (m+n)t}{T} \right) \right] \\
\sin \left( \frac{2\pi mt}{T} \right) \sin \left( \frac{2\pi nt}{T} \right) &= \frac{1}{2} \left[ \cos \left( \frac{2\pi (m-n)t}{T} \right) - \cos \left( \frac{2\pi (m+n)t}{T} \right) \right] \\
\cos \left( \frac{2\pi mt}{T} \right) \sin \left( \frac{2\pi nt}{T} \right) &= \frac{1}{2} \left[ \sin \left( \frac{2\pi (m-n)t}{T} \right) - \sin \left( \frac{2\pi (m+n)t}{T} \right) \right] \\
\sin \left( \frac{2\pi mt}{T} \right) \cos \left( \frac{2\pi nt}{T} \right) &= \frac{1}{2} \left[ \sin \left( \frac{2\pi (m-n)t}{T} \right) + \sin \left( \frac{2\pi (m+n)t}{T} \right) \right]
\end{align*}
\]

4. \[
\begin{align*}
\int_{-T/2}^{T/2} \cos \left( \frac{2\pi mt}{T} \right) \cos \left( \frac{2\pi nt}{T} \right) \, dt &= \delta_{mn} \frac{T}{2} \\
\int_{-T/2}^{T/2} \sin \left( \frac{2\pi mt}{T} \right) \sin \left( \frac{2\pi nt}{T} \right) \, dt &= \delta_{mn} \frac{T}{2} \\
\int_{-T/2}^{T/2} \cos \left( \frac{2\pi mt}{T} \right) \sin \left( \frac{2\pi nt}{T} \right) \, dt &= 0 \\
\int_{-T/2}^{T/2} \sin \left( \frac{2\pi mt}{T} \right) \cos \left( \frac{2\pi nt}{T} \right) \, dt &= 0
\end{align*}
\]

5. \[
\begin{align*}
\int_{-T/2}^{T/2} h(t) \cos \left( \frac{2\pi nt}{T} \right) \, dt &= a_n \frac{T}{2} \\
\int_{-T/2}^{T/2} h(t) \sin \left( \frac{2\pi nt}{T} \right) \, dt &= b_n \frac{T}{2} \\
\end{align*}
\]

\[
\begin{align*}
a_n &= 2 \frac{T}{T} \int_{-T/2}^{T/2} h(t) \cos \left( \frac{2\pi nt}{T} \right) \, dt \\
b_n &= 2 \frac{T}{T} \int_{-T/2}^{T/2} h(t) \sin \left( \frac{2\pi nt}{T} \right) \, dt
\end{align*}
\]

3
2 Complex Fourier Series

Putting aside for the moment your previous results with trigonometric Fourier series, consider once again a function \( h(t) \) defined for \(-\frac{T}{2} < t < \frac{T}{2}\). Now assume it can be described by the expansion

\[
h(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(-i \frac{2\pi nt}{T}\right)
\]

(exp(x) is just another way of writing \( e^x \) which is more legible if \( x \) is a complicated expression.) Note that now \( n \) ranges over negative as well as positive integers.

1. Write the complex conjugate \( h^*(t) \). By redefining the summation index (e.g., to be \( m = -n \)), write \( h^*(t) \) as a complex Fourier series with the same exponentials and different coefficients. If \( h(t) \) is a real function (\( h^*(t) = h(t) \)), what conditions must the coefficients \( \{c_n|n = -\infty, \ldots, \infty\} \) satisfy? What if it’s an imaginary function (\( h^*(t) = -h(t) \))? What can you say about \( c_0 \) in each case?

2. Write \( h(-t) \), once again changing the index to express it as a Fourier series with different coefficients. If \( h(t) \) is an even function (\( h(-t) = h(t) \)), what conditions does this set on the coefficients \( \{c_n|n = -\infty, \ldots, \infty\} \)? What if it’s an odd function (\( h(-t) = -h(t) \))? How does this differ from the conditions in the previous exercise?

3. Apply the rule \( e^\alpha e^\beta = e^{\alpha+\beta} \) for products of exponentials to obtain an expression for

\[
\exp\left(i \frac{2\pi mt}{T}\right) \exp\left(-i \frac{2\pi nt}{T}\right)
\]

4. Using the Euler relation \( e^{i\theta} = \cos \theta + i \sin \theta \), calculate \( e^{i2\pi} \). Use this to obtain an expression for \( e^{i(\theta+2\pi N)} \), where \( N \) is any integer.

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\( ^2 \)A shortcut to taking complex conjugates is to change every \( i \) you see to \(-i\) and to put a star on every complex parameter or variable you see. So for instance, if \( z \) is complex and \( \theta \) is real, \((ze^{i\theta})^* = z^*e^{-i\theta}\)
5. Using the last two results, calculate the integrals

\[
\int_{-T/2}^{T/2} \exp \left( i \frac{2\pi mt}{T} \right) \exp \left( -i \frac{2\pi nt}{T} \right) dt \quad m \neq n
\]

\[
\int_{-T/2}^{T/2} \exp \left( i \frac{2\pi mt}{T} \right) \exp \left( -i \frac{2\pi nt}{T} \right) dt \quad m = n
\]

where \( m \) and \( n \) are integers.

Combine both results into a single expression for

\[
\int_{-T/2}^{T/2} \exp \left( i \frac{2\pi mt}{T} \right) \exp \left( -i \frac{2\pi nt}{T} \right) dt
\]

using the “Kronecker delta”

\[
\delta_{mn} = \begin{cases} 
1 & m = n \\
0 & m \neq n 
\end{cases}
\]

6. Consider the sum

\[
\sum_{n=-\infty}^{\infty} A_n \delta_{mn}
\]

If \( m \) is an integer, there must be one term in the sum where \( n = m \). What is the value of this term? What is the value of any term with \( n \neq m \)? Use the results to obtain a simple expression for the entire sum, assuming \( m \) is an integer.

7. Use the previous results to calculate the integral

\[
\int_{-T/2}^{T/2} h(t) \exp \left( i \frac{2\pi mt}{T} \right) dt
\]

where \( m \) is any integer. Use this result to obtain an expression for \( c_m \).

8. Suppose the same function \( h(t) \) can be written as a complex exponential Fourier series and a trigonometric Fourier series:

\[
h(t) = \sum_{n=-\infty}^{\infty} c_n \exp \left( -i \frac{2\pi nt}{T} \right)
\]

\[
= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{2\pi nt}{T} \right) + \sum_{n=1}^{\infty} b_n \sin \left( \frac{2\pi nt}{T} \right)
\]

Using the Euler relation \( e^{i\theta} = \cos \theta + i \sin \theta \), rewrite the first expression as a sum of sines and cosines, and by matching up coefficients obtain expressions for \( a_0 \), \( \{a_n|n = 0, \ldots \infty\} \) and \( \{b_n|n = 1, \ldots \infty\} \) in terms of \( \{c_n|n = -\infty, \ldots \infty\} \).
2.1 Answers

1. 

\[ h^*(t) = \sum_{n=\infty}^{\infty} c_n^* \exp \left( \frac{2\pi nt}{T} \right) \]

\[ = \sum_{n=\infty}^{\infty} c_n^* \exp \left( -\frac{2\pi nt}{T} \right) \]

If \( h(t) \) is real, we must have \( c_n = c_n^* \) for all integer \( n \). In particular, this means \( c_0 = c_0^* \), or \( c_0 \) is real. If \( h(t) \) is imaginary, we must have \( c_n = -c_n^* \) for all integer \( n \). In particular, this means \( c_0 = -c_0^* \), or \( c_0 \) is imaginary.

2. 

\[ h(-t) = \sum_{n=\infty}^{\infty} c_n \exp \left( \frac{2\pi nt}{T} \right) \]

\[ = \sum_{n=\infty}^{\infty} c_{-n} \exp \left( -\frac{2\pi nt}{T} \right) \]

If \( h(t) \) is even, we must have \( c_n = c_{-n} \) for all integer \( n \). If \( h(t) \) is odd, we must have \( c_n = -c_{-n} \) for all integer \( n \). These conditions differ from the previous one in that they don’t involve the complex conjugate.

3. 

\[ \exp \left( \frac{2\pi mt}{T} \right) \exp \left( -\frac{2\pi nt}{T} \right) = \exp \left( \frac{2\pi (m-n)t}{T} \right) \]

4. \( e^{i2\pi} = 1 \), so \( e^{i(\theta+2\pi N)} = e^{i\theta} \).

5. 

\[ \int_{-T/2}^{T/2} \exp \left( \frac{2\pi mt}{T} \right) \exp \left( -\frac{2\pi nt}{T} \right) dt = T \delta_{mn} \]

6. When \( n = m \), \( A_n \delta_{mn} = A_m \), and when \( n \neq m \), \( A_n \delta_{mn} = 0 \), so

\[ \sum_{n=-\infty}^{\infty} A_n \delta_{mn} = A_m \]

7. 

\[ \int_{-T/2}^{T/2} h(t) \exp \left( \frac{2\pi mt}{T} \right) dt = T c_m \]

so

\[ c_m = \frac{1}{T} \int_{-T/2}^{T/2} h(t) \exp \left( \frac{2\pi mt}{T} \right) dt \]

8. 

\[ a_0 = 2 c_0 \]

\[ a_n = c_n + c_{-n} \quad n = 1, \ldots, \infty \]

\[ b_n = \frac{c_n - c_{-n}}{i} \quad n = 1, \ldots, \infty \]