Simulating Relativistic Perfect Fluids

Scott C. Noble

with

M. W. Choptuik

January 18, 2001

Investigations

- Collapse of Neutron (TOV) Star Perturbed via Minimally-Coupled EMKG Field
  - Can NS be driven to a critical sol'n?
  - Do dispersals leave static remnants?

- Accelerating Blast Waves (w/o gravity)
  - Rel. shock through low ρ atmosphere;
  - Used in Fireball models of GRB's;

- 1D Code as Test Bed for 2D/3D (Stellar Collapse, Accretion Studies, ...);

Relativistic Perfect Fluids

- Continuous Matter Distribution;
- No heat conduction;
- No shear stresses or viscosity;

$$T_{ab} = (\rho + P)\, u_a u_b + P g_{ab}$$

$$J^a = \rho_\circ u^a$$

where

$J^a =$ matter current;

$\rho_\circ =$ rest mass density;

$\epsilon =$ specific internal energy;

$\rho = (1 + \epsilon)\, \rho_\circ =$ energy density;

$P =$ pressure;

$u^a =$ time-like 4-velocity $(u_a u^a = -1)$;

Equations of Motion:

- Local Conservation of Energy : $\nabla_a T^a{}_b = 0$
- Local Conservation of Current : $\nabla_a J^a = 0$

Equations of State :

- Synge Gas (Polytropic) :

$$P(\rho_\circ, \epsilon) = (\Gamma - 1)\,\rho_\circ \epsilon$$

  - Describes Monatomic Nondegenerate Rel. Fluids (Rel. Perfect Gas Law)

- Ultrarelativistic (Barotropic), Self-Similar :

$$P(\rho) = (\Gamma - 1)\,\rho$$

  - Ex] : For, $\Gamma = 4/3$:
    - Fluid of e^+/e^- pairs;
    - Fluid of massless neutrinos;
    - Radiation dominated fluid;

  - When $\rho_\circ \epsilon \gg \rho_\circ \Longrightarrow \rho \simeq \rho_\circ \epsilon$
    Synge Gas EOS $\simeq$ Ultrarel. EOS

Previous Work

Founders:

- May and White (1966)
  – Stellar Collapse, 1D Lagrangian Code (w/ AV);
- Wilson (1972)
  – 1D-3D Eulerian Formulation (w/ AV);
- U Valencia: Ibáñez, Martí, Romero (no AV);
  – Intro. of HRSC Conservative Methods to GR;

Current Work

Relativistic Fluids w/ HRSC Conservative Schemes:

- Wash U / AEI / NCSA :
  (NS/NS, AMR via Cactus);
- Osaka U / UICU : Shibata, Shapiro, et al. :
  (NS/NS, no AMR);
- Drexel U $\rightarrow$ Goddard-NASA : Centrella, et al.
  ($\approx$ NS/NS, "Pseudo-AMR");
- UBC / UT : Choptuik, Neilsen, Noble;
  (1D Crit. Phen., 2D w/ scalar gravity, $\approx$ AMR)
- U Valencia ;

Critical Phenomenon of Perfect Fluids:

- **Evans and Coleman (1994) :**
  - $P = \rho/3$
  - Found Type II CSS via direct PDE sol'n and by taking CSS ansatz;
- **Koike, et al. (1995) :**
  - $P = \rho/3$
  - For $M_{BH} \propto |p - p^*|^\gamma$, $\gamma = (\Re_{\mathrm{e}}\ \mathtt{Lyapunov\ Exponent})^{-1}$
- **Maison (1996) :**
  - Taking CSS ansatz w/ $P = (\Gamma - 1)\rho$ that $\gamma = \gamma(\Gamma)$;
  - Looked at $1.01 \leq \Gamma \leq 1.888$
  - $\Gamma \approx 1.89$ was the "limit";
- **Neilsen and Choptuik (1999-2000) :**
  - Used EOM's PDE sol'n and CSS ansatz for $1.01 \leq \Gamma \leq 2$
  - Found Synge gas' Critical Solutions correspond to barotrope's;

Synge Gas Equations , $P = (\Gamma - 1)\rho_\circ \epsilon$:

$$\boxed{\dot{\mathbf{q}} + \tfrac{1}{r^2}\left(r^2 X \mathbf{f}\right)' = \boldsymbol{\psi}}$$

$$\mathbf{q} = \begin{bmatrix} D \\ S \\ \tau \end{bmatrix} \ , \ \mathbf{f} = \begin{bmatrix} Dv \\ Sv + P \\ v\,(\tau + P) \end{bmatrix} \ , \ \boldsymbol{\psi} = \begin{bmatrix} 0 \\ \Sigma \\ 0 \end{bmatrix} \ , \ \mathbf{w} = \begin{bmatrix} \rho_\circ \\ v \\ P \end{bmatrix}$$

$$X \equiv \frac{\alpha}{a} \quad , \quad \Sigma \equiv \Theta + \frac{2PX}{r}$$

$$\Theta \equiv \alpha a \left[(Sv - \tau - D) \left(8\pi r P + \frac{M}{r^2} \right) + P \frac{M}{r^2} \right]$$

$$D = a\rho_\circ W \ , \quad S = (\rho + P)\, W^2 v \ , \quad \tau = S/v - D - P$$

$$v = \frac{a u^r}{\alpha u^t} \quad , \quad W^2 = \frac{1}{1 - v^2}$$

Polar-Areal Coordinates

$$ds^2 = -\alpha\left(r,t\right)^2 dt^2 + a\left(r,t\right)^2 dr^2 + r^2 d\Omega^2$$

Slicing Condition :

$$\frac{\alpha'}{\alpha} = a^2\left[4\pi r\left(Sv + P\right) + \frac{M}{r^2}\right]$$

Hamiltonian Constraint :

$$\frac{a'}{a} = a^2\left[4\pi r\left(\tau + D\right) - \frac{M}{r^2}\right]$$

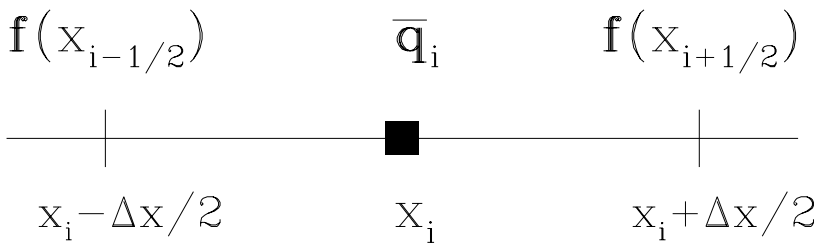
g_{rr} **Evolution :**

$$\dot{a} = -4\pi r\alpha a^2 S$$

Mass Aspect Function : $M = \frac{r}{2}\left(1 - 1/a^2\right)$

Finite Volume Method

- Conservative (finds physical solutions);
- Can handle discontinuities (shocks);



Take $\partial_t \mathbf{q} + \partial_x \mathbf{f} = 0$

- $\int_{-\infty}^{\infty} q(x,t)dx =$ a conserved quantity
- $f(x,t) = f(q(x,t))$ flux of q at x,t
- Cells $\mathcal{C}_i$ centered at x_i with boundaries at $x_i - \Delta x/2$, $x_i + \Delta x/2$.

$$\bar{q}_i(t) = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)dx$$

$$\frac{d}{dt}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)dx = f\left(q(x_{i-1/2},t)\right) - f\left(q(x_{i+1/2},t)\right)$$

$$\Delta x \left(\bar{q}_i(t_2) - \bar{q}_i(t_1)\right) = \int_{t_1}^{t_2} f\left(q(x_{i-1/2}, t)\right) dt - \int_{t_1}^{t_2} f\left(q(x_{i+1/2}, t)\right) dt$$

- Define Numerical Flux, F, as "time average" of f:

$$F_i^n = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f\left(q(x_i, t)\right) dt$$

$$\bar{q}_i^{n+1} = \bar{q}_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$

where $t^n = (n-1)\Delta t$;

Riemann Problem

- Find $\mathbf{q}(x,t)$ given

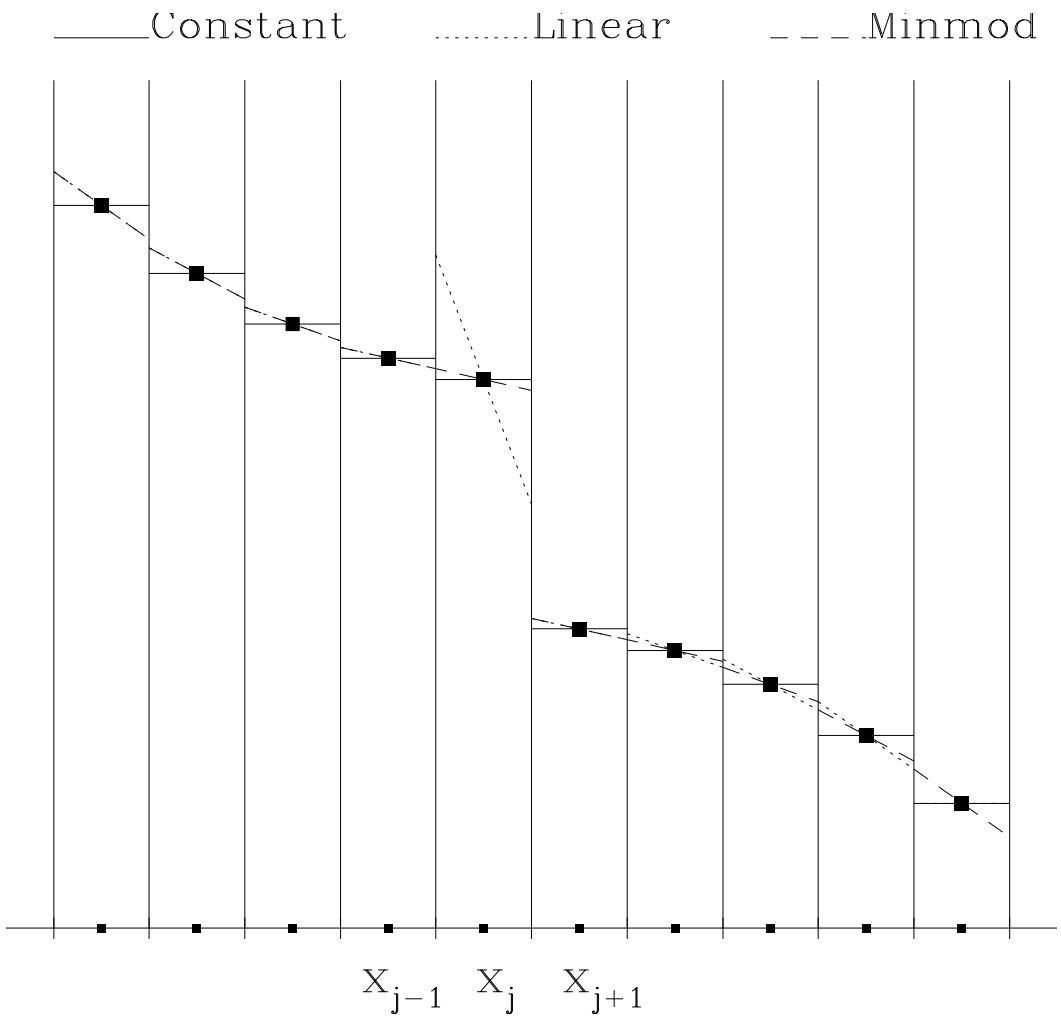
$$\partial_t \mathbf{q} + \partial_x \mathbf{f} = 0$$

$$\mathbf{q}(x, t=0) \;=\; \begin{cases} \mathbf{q}_L & \text{for } x < 0 \\ \mathbf{q}_R & \text{for } x > 0 \end{cases}$$

- Sol'n via characteristics of quasi-linear form

$$\mathbf{q}_t + \mathbf{A}\mathbf{q}_x = 0 \quad , \quad \mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{q}}$$

and via Rankine-Hugoniot jump conditions, which tell how discontinuities propagate.



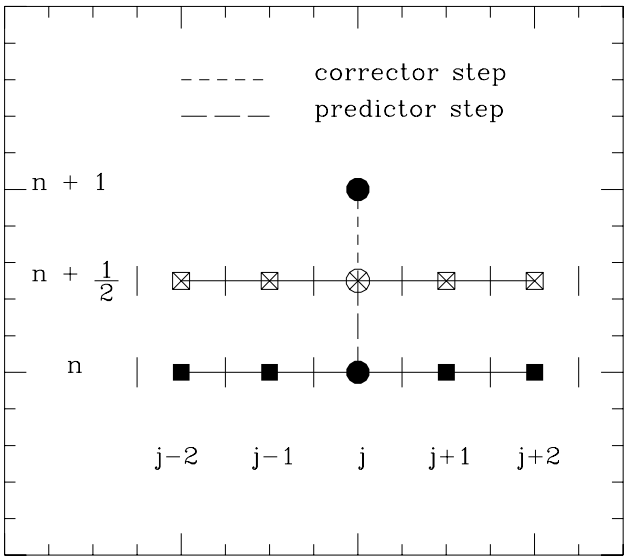
At $x = x_{j+1/2}$	Const.	Linear	Minmod
$\tilde{\mathbf{q}}_L =$	$\bar{\mathbf{q}}_j$	$\bar{\mathbf{q}}_j + \mathbf{s}_{j+1/2}\Delta x/2$	$\bar{\mathbf{q}}_j + \mathrm{M}(\mathbf{s}_{j+1/2}, \mathbf{s}_{j-1/2})\Delta x/2$
$\tilde{\mathbf{q}}_R =$	$\bar{\mathbf{q}}_{j+1}$	$\bar{\mathbf{q}}_{j+1} - \mathbf{s}_{j+3/2}\Delta x/2$	$\bar{\mathbf{q}}_{j+1} - \mathrm{M}(\mathbf{s}_{j+3/2}, \mathbf{s}_{j+1/2})\Delta x/2$

$$\mathbf{s}_{j+1/2} \equiv \left(\bar{\mathbf{q}}_{j+1} - \bar{\mathbf{q}}_j\right)/\Delta x$$

$$\mathrm{M}(a,b) = \mathrm{Minmod}(a,b) = \begin{cases} 0 & \text{if} \quad ab < 0 \\ a & \text{if} \quad |a| < |b| \quad \text{and} \quad ab > 0 \\ b & \text{if} \quad |b| < |a| \quad \text{and} \quad ab > 0 \end{cases}$$

Integration Scheme

- Method of Lines;
- Midpoint Predictor-Corrector Method :
  - a.k.a 2nd-order Runge-Kutta method;



- If $\partial_t \mathbf{q} = L(\mathbf{q})$:

$$\bar{\mathbf{q}}^{n+1/2} = \bar{\mathbf{q}}^{n} + \frac{\Delta t}{2}\hat{L}(\bar{\mathbf{q}}^{n})$$

$$\bar{\mathbf{q}}^{n+1} = \bar{\mathbf{q}}^{n} + \Delta t \hat{L}\left(\bar{\mathbf{q}}^{n+1/2}\right)$$

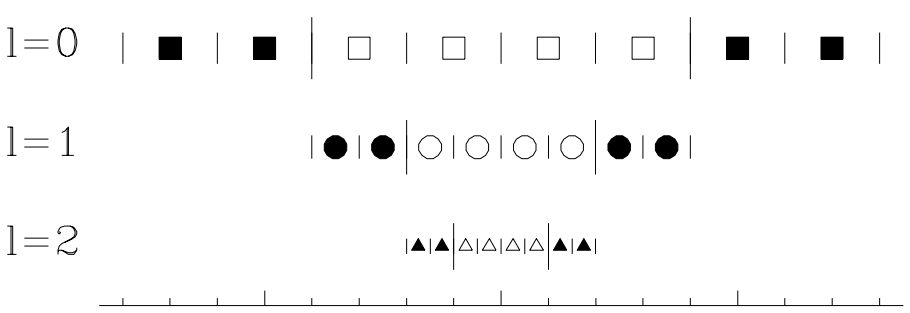
$$\hat{L} = -\frac{3\left[\left(r^2 X \mathbf{F}(\bar{\mathbf{q}})\right)_{i+1/2} - \left(r^2 X \mathbf{F}(\bar{\mathbf{q}})\right)_{i-1/2}\right]}{r^3_{i+1/2} - r^3_{i-1/2}} + \boldsymbol{\psi}_i$$

Intro. to Adaptive Mesh Refinement

- Berger & Oliger, 1983;

- Efficiently Resolves Detail;
  - Greatly needed for multi-dimensions;
  - Automatic – Follows Features;

- A Grid, G_j^l, is a collection of cells (points);
  - $\Delta r^l = R\ \Delta r^{l+1}$, $\Delta t^l = R\ \Delta t^{l+1}$,
  - $G^l = \bigcup_j G_j^l$, G^0 coarsest
  - Grid functions found $\forall$ cells, $\forall\ G_j^l$;
  - Local Truncation Error (LTE) decides regridding;



AMR and Fluids

- Previous Work (Classical Fluids):
  - Berger & Colella, 1989;
  - LeVeque;

- Maintain Conservative Nature ;

- Able to Capture Shocks;

- Finite Volume Interpolation/Injection;
  Points $\rightarrow$ Cells

- Need Way to Refine for Method of Lines;

- Use Ghost Cells for Boundary Conditions;

Coarse-To-Fine Interpolation:

- Need to make it shock capture ;
  and conservative;

$$\bar{\mathbf{q}}_k^\ell \;=\; \frac{1}{\Delta r^\ell}\int_{r^\ell_{k-1/2}}^{r^\ell_{k+1/2}} \mathbf{Q}_i^{\ell-1}(r)\,dr$$

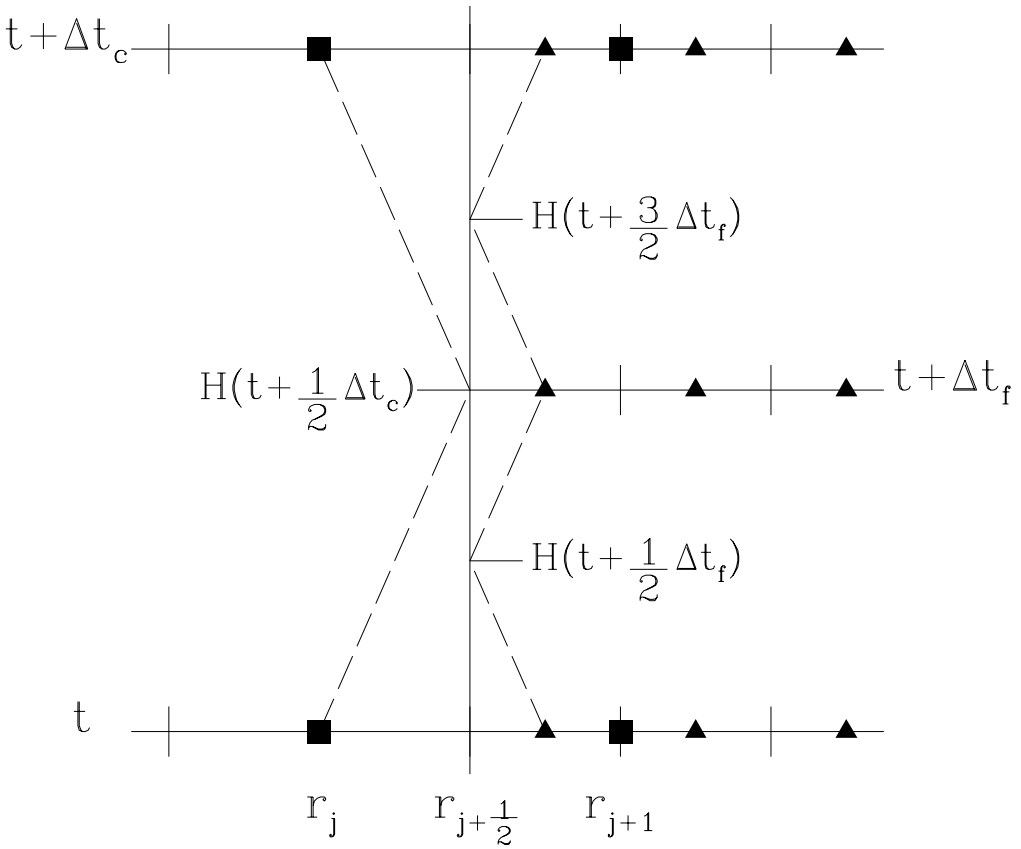
$$\mathbf{Q}_i^{\ell-1}(r) = \sum_{n=0}^{N-1} \mathbf{a}_n \left(r - r_i^\ell\right)^n \quad , \quad \mathbf{a}_n = \mathbf{a}_n\left(\bar{\mathbf{q}}^{\ell-1}\right)$$

Fine-To-Coarse Injection:

- Used to update coarse cells;
- Maintain notion of cell average:

$$\mathbf{q}^l(r_j) \to \frac{1}{R}\sum_{k=1}^{R} \mathbf{q}^{l+1}(r_k)$$

Flux Injector



$$\mathbf{H}_j \equiv \frac{\alpha}{a}\mathbf{F}_j$$

$$\mathbf{H}^{coarse}_{j+1/2}(t^{n+1/2}) \rightarrow \frac{1}{R}\sum_{k=1}^{R} \mathbf{H}^{fine}_{j+1/2}(t^{n+k-1/2})$$

- Then, coarse $\mathbf{q}^{n+1}$ is recalculated;